

ANALYSIS OF A COUPLED SYSTEM OF KINETIC EQUATIONS AND CONSERVATION LAWS: RIGOROUS DERIVATION AND EXISTENCE THEORY VIA DEFECT MEASURES

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ABSTRACT. In this paper we introduce a coupled system of kinetic equations of B.G.K. type and then study its hydrodynamic limit. We obtain as a consequence the rigorous derivation and existence theory for a coupled system of kinetic equations and their hydrodynamic (conservation laws) limit. The latter is a particular case of the coupled system of Boltzmann and Euler equations. A fundamental element in this study is the rigorous derivation and justification of the interface conditions between the kinetic model and its hydrodynamic conservation laws limit, which is obtained using a new regularity theory introduced herein.

1. INTRODUCTION

In this paper we shall study the following coupled kinetic system:

- (1) $[\partial_t + a(v) \cdot \partial_x]f_\epsilon(x, v, t) = \frac{1}{\epsilon_l}(\chi_{u_\epsilon(x,t)}(v) - f_\epsilon(x, v, t))$ in $\Omega_l \times V \times (0, T)$,
- (2) $f_\epsilon(x, v, t) = f_{\epsilon 0}(x, v, t)$ on $\Gamma_1^- \times (0, T)$,
- (3) $f_\epsilon(x, v, t) = g_\epsilon(x, v, t)$ on $\Gamma_i^- \times (0, T)$,
- (4) $[\partial_t + a(v) \cdot \partial_x]g_\epsilon(x, v, t) = \frac{1}{\epsilon}(\chi_{w_\epsilon(x,t)}(v) - g_\epsilon(x, v, t))$ in $\Omega_g \times V \times (0, T)$,
- (5) $g_\epsilon(x, v, t) = g_{\epsilon 1}(x, v, t)$ on $\Gamma_2^- \times (0, T)$,
- (6) $g_\epsilon(x, v, t) = f_\epsilon(x, v, t)$ on $\Gamma_i^+ \times (0, T)$,
- (7) $f_\epsilon(x, v, 0) = f_\epsilon^0(x, v)$ in $\Omega_l \times V$ $g_\epsilon(x, v, 0) = g_\epsilon^0(x, v)$ in $\Omega_g \times V$

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and study its relation to the coupled system

$$(8) \quad [\partial_t + a(v) \cdot \partial_x]f(x, v, t) = \frac{1}{\epsilon_l}(\chi_{u(x,t)}(v) - f(x, v, t)) \text{ in } \Omega_l \times V \times (0, T),$$

$$(9) \quad f(x, v, t) = f_0(x, v, t) \text{ on } \Gamma_1^- \times (0, T),$$

$$(10) \quad f(x, v, t) = \chi_{w(x,t)}(v) \text{ on } \Gamma_i^- \times (0, T),$$

$$(11) \quad \partial_t w + \partial_{x_i}[A_i(w)] = 0 \text{ in } \Omega_g \times (0, T),$$

$$(12) \quad \text{boundary conditions for } w \text{ on } \Gamma_2 \times (0, T) \text{ and } \Gamma_i \times (0, T),$$

$$(13) \quad f(x, v, 0) = f^0(x, v) \text{ in } \Omega_l \times V \quad w(x, 0) = w^0(x) \text{ in } \Omega_g.$$

Here, $\Omega_l = (0, l) \times \mathbb{R}^{d-1}$, $\Omega_g = (l, 1) \times \mathbb{R}^{d-1}$ with $0 < l < 1$, and $\Omega = (0, 1) \times \mathbb{R}^{d-1}$ are resp. the kinetic, hydrodynamic, and physical domain and d is the dimension of the spatial domain. The boundaries are defined as follows:

$$\begin{aligned} \Gamma_1 &= \partial\Omega_l \cap \partial\Omega, \quad \Gamma_2 = \partial\Omega_g \cap \partial\Omega, \quad \Gamma_i = \partial\Omega_l \cap \partial\Omega_g, \\ \Gamma_1^- &= \{(x, v) \in \{0\} \times \mathbb{R}^{d-1} \times V : a(v) \cdot n_l(x) < 0\}, \\ \Gamma_i^- &= \{(x, v) \in \{l\} \times \mathbb{R}^{d-1} \times V : a(v) \cdot n_l(x) < 0\}, \\ \Gamma_2^- &= \{(x, v) \in \{1\} \times \mathbb{R}^{d-1} \times V : a(v) \cdot n_g(x) < 0\}, \\ \Gamma_{0l} &= \{(x, v) \in \partial\Omega_l \times V : a(v) \cdot n_l(x) = 0\}, \\ \Gamma_{0g} &= \{(x, v) \in \partial\Omega_g \times V : a(v) \cdot n_g(x) = 0\}, \end{aligned}$$

where n_l resp. n_g denotes the exterior unit normal vector to Ω_l resp. Ω_g . In the sequel we will denote by n any exterior unit normal vector; it will be clear from the context which normal we mean. We shall assume that Γ_{0l} has measure 0 in $\partial\Omega_l \times V$ for the measure $d\gamma dv$ and that Γ_{0g} has measure 0 in $\partial\Omega_g \times V$ for the measure $d\gamma dv$.

The boundary conditions in (12) for the conservation laws are prescribed on a part of Γ_i resp. Γ_2 . These boundary conditions together with those in (10) will be made precise in Definition 3.1. The set $V = \mathbb{R}$ is the velocity domain and T is a positive constant.

The function f describes the microscopic density of particles at (x, t) with velocity v in the kinetic domain. The function w describes the local density of particles at (x, t) in the hydrodynamic domain.

The functions f_ϵ and f resp. g_ϵ describe the microscopic density of particles at (x, t) with velocity v in the kinetic resp. hydrodynamic domain. The physical parameter ϵ_l resp. $\epsilon > 0$ is the microscopic scale in the kinetic resp. hydrodynamic domain. The functions f_ϵ^0 and g_ϵ^0 are the initial data while $f_{\epsilon 0}$, f_0 , and $g_{\epsilon 1}$ are the boundary data. The boundary conditions in Eqs. (3) and (6) describe the transmission conditions between the two kinetic equations. The local densities of particles u_ϵ , u , and w_ϵ at (x, t) are related to the microscopic densities f_ϵ , f , and g_ϵ by $u_\epsilon(x, t) = \int_V f_\epsilon(x, v, t) dv$, $u(x, t) = \int_V f(x, v, t) dv$, and $w_\epsilon(x, t) = \int_V g_\epsilon(x, v, t) dv$. The boundary conditions in (12) also involve w_1 which is a given boundary data; see Definition 3.1 below. The boundary conditions in Eqs. (10) and (12) describe the transmission conditions between the kinetic and the hydrodynamic domain. Let $A = (A_i)_{1 \leq i \leq d}$ with components related to $a_i(\cdot)$ by $a_i(\cdot) = A'_i(\cdot)$, $i = 1, \dots, d$. We

assume that the functions $a_i(\cdot) \in C^{2,\alpha}$ for some $\alpha > 0$. The collisions in the kinetic equations are given by the nonlinear kernel in the right-hand side of Eqs. (1), (4), and (8) in which $\chi_u(v)$ is the signature of u defined by

$$(14) \quad \chi_u(v) = \begin{cases} 1 & \text{if } 0 < v \leq u, \\ -1 & \text{if } u \leq v < 0, \\ 0 & \text{otherwise.} \end{cases}$$

This problem is a particular case of the more general coupled systems consisting of Boltzmann equations and their hydrodynamic limits introduced and studied in [21, 22, 23] (see also the references therein). These problems are introduced in order to solve the problem of intermediate regimes in gas dynamics. Their practical importance in real life applications has been successfully proven in [21, 22, 23] (see also the references therein). A complete mathematical theory of these methodologies in the case of a Carleman simplified Boltzmann model has been developed by the author in [23]. Here we shall study a nonlinear Boltzmann model which is a bit more general. The study of the hydrodynamic limit of the model in (4) in full space was done by Perthame and Tadmor [16], Perthame [15], and the author [26, 27]. They proved that this model converges as the microscopic scale goes to 0 to a conservation law of the form in (11). In the case of bounded domain such study is performed by the author in [26, 27]. Further studies using this model and closely related models can be found in the references in [15].

Here we study this nonlinear model in the context of domains with boundaries and coupling of Boltzmann equations and their hydrodynamic limits. More precisely we shall study the hydrodynamic limit of the kinetic system (1)-(7) as the microscopic scale $\epsilon \rightarrow 0$ in Ω_g . As a consequence we shall rigorously derive and establish the existence theory of the system (8)-(13). Systems (1)-(7) have been introduced by the author in [25], where he established the hydrodynamic limit analysis of this kinetic system and rigorously derived system (8)-(13). In [25], the author made the additional assumption that the initial data are in BV_{loc} . Here we shall remove such an assumption, however, we shall make a nondegeneracy assumption on the flux.

A fundamental element in this study is the rigorous derivation of the interface conditions between the kinetic model and hydrodynamic conservation laws limit. This is obtained using a new regularity theory developed here. Besides its intrinsic merit, we expect that the study developed here of the interfacial kinetic-hydrodynamic problem may provide methods and ideas that might help solve other problems involving interfaces between physical phenomena of a different scale and nature.

In the next section we shall study the kinetic system and give various information about its solution. These results have been proved in [25]. We shall also give some results about generalized Green and Gauss-Green formulas for bounded Radon measures and about the weak traces for a particular class of functions. In Section 3, we shall introduce the kinetic formulation of the coupled system (8)-(13) and prove its equivalence to system (8)-(13) in the sense of Definitions 3.1 and 3.2. In the fourth section we prove that the solution of the kinetic system converges to a solution of the system given by the kinetic formulation. Finally, in the last section we prove that the system (8)-(13) has an entropic solution in the sense of Definition 3.1.

2. COUPLED SYSTEM OF KINETIC EQUATIONS

In this section we shall present some basic results about the kinetic system (1)-(7). These results and their proofs were obtained in [25]. However, for the sake of completeness and independent reading we will give below the proofs of such results. We begin with stating a result about the well-posedness of the kinetic system (1)-(7). We then establish various properties of the solution, including L^∞ , L^1 , and BV_{loc} uniform estimates. These estimates will be used for the study of the hydrodynamic limit of system (1)-(7) as $\epsilon \rightarrow 0$ in Ω_g . We shall also use the following notations:

$$\begin{aligned}\Omega_0 &= \{(x, v, t) \in \Omega_l \times V \times (0, T) : x_1 - a_1(v)t < 0\}, \\ \Omega_a &= \{(x, v, t) \in \Omega_l \times V \times (0, T) : x_1 - a_1(v)t > l\}, \\ \Omega_{0a} &= \{(x, v, t) \in \Omega_l \times V \times (0, T) : 0 < x_1 - a_1(v)t < l\}, \\ \Omega_1 &= \{(x, v, t) \in \Omega_g \times V \times (0, T) : x_1 - a_1(v)t > 1\}, \\ \Omega_{a1} &= \{(x, v, t) \in \Omega_g \times V \times (0, T) : l < x_1 - a_1(v)t < 1\}, \\ \Omega_{aa} &= \{(x, v, t) \in \Omega_g \times V \times (0, T) : x_1 - a_1(v)t < l\},\end{aligned}$$

where $x = (x_1, x_\star)$.

2.1. Existence theory. We have the following existence and uniqueness result.

Theorem 2.1. *Assume that*

$$\begin{aligned}f_\epsilon^0 &\in L^1(\Omega_l \times V), \quad g_\epsilon^0 \in L^1(\Omega_g \times V), \\ a(v) \cdot n_l f_{\epsilon 0} &\in L^1(\Gamma_1^- \times (0, T)), \quad a(v) \cdot n_g g_{\epsilon 1} \in L^1(\Gamma_2^- \times (0, T)).\end{aligned}$$

Then

(i) *The coupled system (1)-(7) has a unique solution (f_ϵ, g_ϵ) in $L^\infty([0, T]; L^1(\Omega_l \times V)) \times L^\infty([0, T]; L^1(\Omega_g \times V))$. Moreover, (f_ϵ, g_ϵ) satisfies the integral representation*

$$\begin{aligned}\text{In } \Omega_0 \quad f_\epsilon(x, v, t) &= \exp(-x_1/\epsilon_l a_1(v)) f_\epsilon((0, x_\star - \frac{x_1}{a_1(v)} a_\star(v)), v, t - x_1/a_1(v)) \\ &\quad + \frac{1}{\epsilon_l} \int_{t-x_1/a_1(v)}^t e^{(s-t)/\epsilon_l} \chi_{u_\epsilon(x(s), s)}(v) ds. \\ \text{In } \Omega_a \quad f_\epsilon(x, v, t) &= g_\epsilon((l, x_\star + \frac{l-x_1}{a_1(v)} a_\star(v)), v, t - \frac{x_1-l}{a_1(v)}) \exp(-(x_1-l)/\epsilon_l a_1(v)) \\ &\quad + \frac{1}{\epsilon_l} \int_{t-\frac{x_1-l}{a_1(v)}}^t e^{(s-t)/\epsilon_l} \chi_{u_\epsilon(x(s), s)}(v) ds. \\ \text{In } \Omega_{0a} \quad f_\epsilon(x, v, t) &= f_\epsilon^0(x - a(v)t, v) \exp(-t/\epsilon_l) + \frac{1}{\epsilon_l} \int_0^t e^{(s-t)/\epsilon_l} \chi_{u_\epsilon(x(s), s)}(v) ds. \\ \text{In } \Omega_{aa} \quad g_\epsilon(x, v, t) &= f_\epsilon((l, x_\star + \frac{l-x_1}{a_1(v)} a_\star(v)), v, t - \frac{x_1-l}{a_1(v)}) \exp(-(x_1-l)/(a_1(v)\epsilon)) \\ &\quad + \frac{1}{\epsilon} \int_{t-\frac{x_1-l}{a_1(v)}}^t e^{(s-t)/\epsilon} \chi_{w_\epsilon(x(s), s)}(v) ds.\end{aligned}$$

$$\text{In } \Omega_{a1} \quad g_\epsilon(x, v, t) = g_\epsilon^0(x - a(v)t, v) \exp(-t/\epsilon) + \frac{1}{\epsilon} \int_0^t e^{(s-t)/\epsilon} \chi_{w_\epsilon(x(s), s)}(v) ds.$$

$$\begin{aligned} \text{In } \Omega_1 \quad g_\epsilon(x, v, t) &= g_{\epsilon 1}((1, x_\star + \frac{1-x_1}{a_1(v)} a_\star(v)), v, t - \frac{x_1-1}{a_1(v)}) \exp((1-x_1)/\epsilon a_1(v)) \\ &+ \frac{1}{\epsilon} \int_{t-\frac{x_1-1}{a_1(v)}}^t e^{(s-t)/\epsilon} \chi_{w_\epsilon(x(s), s)}(v) ds, \end{aligned}$$

where $x(s) = x + (s-t)a(v)$, $x = (x_1, x_\star)$, and $a(v) = (a_1(v), a_\star(v))$.

On $\Gamma_i^- \times (0, T)$, we have:

For $l < l - ta_1(v) < 1$,

$$\begin{aligned} g_\epsilon((l, x_\star), v, t) &= e^{-t/\epsilon} g_\epsilon^0((l - ta_1(v), x_\star - ta_\star(v)), v) \\ &+ \frac{1}{\epsilon} \int_0^t e^{(s-t)/\epsilon} \chi_{w_\epsilon(x(s), s)}(v) ds. \end{aligned}$$

For $1 < l - ta_1(v)$,

$$\begin{aligned} g_\epsilon((l, x_\star), v, t) &= e^{(1-l)/(\epsilon a_1(v))} g_\epsilon((1, x_\star + \frac{1-l}{a_1(v)} a_\star(v)), v, t + \frac{1-l}{a_1(v)}) \\ &+ \frac{1}{\epsilon} \int_{t+\frac{1-l}{a_1(v)}}^t e^{(s-t)/\epsilon} \chi_{w_\epsilon(x(s), s)}(v) ds. \end{aligned}$$

We also have a similar formula for f_ϵ on $\Gamma_i^+ \times (0, T)$.

(ii) Let (f_ϵ, g_ϵ) and (F_ϵ, G_ϵ) be two solutions of (1)-(7) with corresponding densities $u_\epsilon(x, t) = \int_V f_\epsilon(x, v, t) dv$, $w_\epsilon(x, t) = \int_V g_\epsilon(x, v, t) dv$ and $U_\epsilon(x, t) = \int_V F_\epsilon(x, v, t) dv$, $W_\epsilon(x, t) = \int_V G_\epsilon(x, v, t) dv$; and let $f_\epsilon^0, g_\epsilon^0, f_{\epsilon 0}, g_{\epsilon 1}$ resp. $F_\epsilon^0, G_\epsilon^0, F_{\epsilon 0}, G_{\epsilon 1}$ denote the corresponding data. Then for any $0 \leq t \leq T$, we have

$$\begin{aligned} &\|f_\epsilon - F_\epsilon\|_{L^1(\Omega_l \times V)} + \|g_\epsilon - G_\epsilon\|_{L^1(\Omega_g \times V)} + \|a(v) \cdot n(f_\epsilon - F_\epsilon)\|_{L^1(\Gamma_1^+ \times (0, t))} \\ &+ \|a(v) \cdot n(g_\epsilon - G_\epsilon)\|_{L^1(\Gamma_2^+ \times (0, t))} \\ &\leq \|f_\epsilon^0 - F_\epsilon^0\|_{L^1(\Omega_l \times V)} + \|g_\epsilon^0 - G_\epsilon^0\|_{L^1(\Omega_g \times V)} + \|a(v) \cdot n(f_{\epsilon 0} - F_{\epsilon 0})\|_{L^1(\Gamma_1^- \times (0, t))} \\ (15) \quad &+ \|a(v) \cdot n(g_{\epsilon 1} - G_{\epsilon 1})\|_{L^1(\Gamma_2^- \times (0, t))}. \end{aligned}$$

Proof of Theorem 2.1. We begin by proving (ii), which states a result about the uniqueness and the continuous dependence of the solution on the data. The idea of the proof is to use a combination of the author's method [20, 21] and ideas from [10].

The function F_ϵ satisfies an equation similar to Eq. (1). Subtracting this equation from Eq. (1), and multiplying the resulting equation by

$$\varphi_1 = \text{sgn}^\mu(f_\epsilon - F_\epsilon) \psi_1(x, t)$$

with $x \text{sgn}^\mu(x) \geq 0$, $x \in \mathbb{R}$, where ψ_1 is a nonnegative test function and sgn^μ is a regularization of sign function, and integrating by parts, and letting $\mu \rightarrow 0$, we

obtain

$$\begin{aligned}
& \int_{\Omega_l \times V} (|f_\epsilon - F_\epsilon| \psi_1)(\cdot, \cdot, t) + \int_{\Gamma_i^- \times (0, t)} a(v) \cdot n |f_\epsilon - F_\epsilon| \psi_1 \\
& + \int_{\Gamma_i^+ \times (0, t)} a(v) \cdot n |f_\epsilon - F_\epsilon| \psi_1 + \int_{\Gamma_1^+ \times (0, t)} a(v) \cdot n |f_\epsilon - F_\epsilon| \psi_1 \\
& + \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, t)} |f_\epsilon - F_\epsilon| \psi_1 \\
& = \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, t)} (\chi_{u_\epsilon} - \chi_{U_\epsilon}) \operatorname{sgn}(f_\epsilon - F_\epsilon) \psi_1 + \int_{\Omega_l \times V} (|f_\epsilon - F_\epsilon| \psi_1)(\cdot, \cdot, 0) \\
(16) \quad & - \int_{\Gamma_1^- \times (0, t)} a(v) \cdot n |f_{\epsilon 0} - F_{\epsilon 0}| \psi_1 + \int_{\Omega_l \times V \times (0, t)} (\partial_t + a(v) \cdot \partial_x)(\psi_1) |f_\epsilon - F_\epsilon|.
\end{aligned}$$

Using the properties of χ , this yields

$$\begin{aligned}
& \int_{\Omega_l \times V} (|f_\epsilon - F_\epsilon| \psi_1)(\cdot, \cdot, t) + \int_{\Gamma_i^- \times (0, t)} a(v) \cdot n |f_\epsilon - F_\epsilon| \psi_1 \\
& + \int_{\Gamma_i^+ \times (0, t)} a(v) \cdot n |f_\epsilon - F_\epsilon| \psi_1 + \int_{\Gamma_1^+ \times (0, t)} a(v) \cdot n |f_\epsilon - F_\epsilon| \psi_1 \\
& \leq \int_{\Omega_l \times V} (|f_\epsilon - F_\epsilon| \psi_1)(\cdot, \cdot, 0) - \int_{\Gamma_1^- \times (0, t)} a(v) \cdot n |f_{\epsilon 0} - F_{\epsilon 0}| \psi_1 \\
(17) \quad & + \int_{\Omega_l \times V \times (0, t)} (\partial_t + a(v) \cdot \partial_x)(\psi_1) |f_\epsilon - F_\epsilon|.
\end{aligned}$$

Similarly, multiplying Eq. (4) by $\varphi_2 = \operatorname{sgn}^\mu(g_\epsilon - G_\epsilon) \psi_2(x, t)$ with ψ_2 a non-negative test function in $C^1(\bar{\Omega}_g \times [0, T])$, and proceeding as above, we obtain an inequation for $g_\epsilon - G_\epsilon$ similar to Eq. (17). Adding this inequation to Eq. (17), taking ψ_1 and ψ_2 such that $\psi_1 = \psi_2$ on $\Gamma_i \times (0, T)$, and using the transmission conditions, we obtain

$$\begin{aligned}
& \int_{\Omega_l \times V} (|f_\epsilon - F_\epsilon| \psi_1)(\cdot, \cdot, t) + \int_{\Omega_g \times V} (|g_\epsilon - G_\epsilon| \psi_2)(\cdot, \cdot, t) \\
& + \int_{\Gamma_1^+ \times (0, t)} a(v) \cdot n |f_\epsilon - F_\epsilon| \psi_1 + \int_{\Gamma_2^+ \times (0, t)} a(v) \cdot n |g_\epsilon - G_\epsilon| \psi_2 \\
& \leq \int_{\Omega_l \times V} (|f_\epsilon - F_\epsilon| \psi_1)(\cdot, \cdot, 0) + \int_{\Omega_g \times V} (|g_\epsilon - G_\epsilon| \psi_2)(\cdot, \cdot, 0) \\
& - \int_{\Gamma_1^- \times (0, t)} a(v) \cdot n |f_{\epsilon 0} - F_{\epsilon 0}| \psi_1 + \int_{\Omega_l \times V \times (0, t)} (\partial_t + a(v) \cdot \partial_x)(\psi_1) |f_\epsilon - F_\epsilon| \\
(18) \quad & - \int_{\Gamma_2^- \times (0, t)} a(v) \cdot n |g_{\epsilon 1} - G_{\epsilon 1}| \psi_2 + \int_{\Omega_g \times V \times (0, t)} (\partial_t + a(v) \cdot \partial_x)(\psi_2) |g_\epsilon - G_\epsilon|.
\end{aligned}$$

Now taking $\psi_1(t) = \psi_2(t) = 1$ yields the estimate (15).

To prove the existence of a solution to the kinetic system, we introduce the following iterations:

$$\begin{aligned} [\partial_t + a(v) \cdot \partial_x] f_\epsilon^{n+1}(x, v, t) &= \frac{1}{\epsilon_l} (\chi_{u_\epsilon^n(x, t)}(v) - f_\epsilon^{n+1}(x, v, t)) \quad \text{in } \Omega_l \times V \times (0, T), \\ f_\epsilon^{n+1}(x, v, t) &= f_{\epsilon 0}(x, v, t) \quad \text{on } \Gamma_1^- \times (0, T), \\ f_\epsilon^{n+1}(x, v, t) &= g_\epsilon^{n+1}(x, v, t) \quad \text{on } \Gamma_i^- \times (0, T), \end{aligned}$$

$$\begin{aligned} [\partial_t + a(v) \cdot \partial_x] g_\epsilon^{n+1}(x, v, t) &= \frac{1}{\epsilon} (\chi_{w_\epsilon^{n+1}(x, t)}(v) - g_\epsilon^{n+1}(x, v, t)) \quad \text{in } \Omega_g \times V \times (0, T), \\ g_\epsilon^{n+1}(x, v, t) &= g_{\epsilon 0}(x, v, t) \quad \text{on } \Gamma_2^- \times (0, T), \\ g_\epsilon^{n+1}(x, v, t) &= f_\epsilon^{n+1}(x, v, t) \quad \text{on } \Gamma_i^+ \times (0, T), \end{aligned}$$

$$f_\epsilon^{n+1}(x, v, 0) = f_\epsilon^0(x, v) \quad g_\epsilon^{n+1}(x, v, 0) = g_\epsilon^0(x, v).$$

Using (16) in the present context with $f_\epsilon = f_\epsilon^{n+1}$ and $F_\epsilon = f_\epsilon^{m+1}$, and the properties of χ , we obtain

$$\begin{aligned} & \int_{\Omega_l \times V} (|f_\epsilon - F_\epsilon| \psi_1)(\cdot, \cdot, t) + \int_{\Gamma_i^- \times (0, t)} a(v) \cdot n |f_\epsilon - F_\epsilon| \psi_1 \\ & + \int_{\Gamma_i^+ \times (0, t)} a(v) \cdot n |f_\epsilon - F_\epsilon| \psi_1 \\ & + \int_{\Gamma_1^+ \times (0, t)} a(v) \cdot n |f_\epsilon - F_\epsilon| \psi_1 + \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, t)} |f_\epsilon - F_\epsilon| \psi_1 \\ (19) \quad & \leq \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, t)} |f_\epsilon^n - f_\epsilon^m| \psi_1 + \int_{\Omega_l \times V \times (0, t)} (\partial_t + a(v) \cdot \partial_x)(\psi_1) |f_\epsilon - F_\epsilon|. \end{aligned}$$

Similarly taking $g_\epsilon = g_\epsilon^{n+1}$ and $G_\epsilon = g_\epsilon^{m+1}$, and using the properties of χ , we obtain

$$\begin{aligned} & \int_{\Omega_g \times V} (|g_\epsilon - G_\epsilon| \psi_2)(\cdot, \cdot, t) + \int_{\Gamma_i^+ \times (0, t)} a(v) \cdot n_g |g_\epsilon - G_\epsilon| \psi_2 \\ & + \int_{\Gamma_i^- \times (0, t)} a(v) \cdot n_g |g_\epsilon - G_\epsilon| \psi_2 + \int_{\Gamma_2^+ \times (0, t)} a(v) \cdot n_g |g_\epsilon - G_\epsilon| \psi_2 \\ (20) \quad & \leq \int_{\Omega_g \times V \times (0, t)} (\partial_t + a(v) \cdot \partial_x)(\psi_2) |g_\epsilon - G_\epsilon|. \end{aligned}$$

Taking $\psi_1 = \psi_2 = e^{-\frac{\alpha}{\epsilon_l} s}$, $0 \leq s \leq t$, with α a positive constant, adding the equations in (19) and (20), and using the transmission conditions and the properties of χ , we

obtain

$$\begin{aligned}
 & \int_{\Omega_l \times V} (|f_\epsilon - F_\epsilon| \psi_1)(\cdot, \cdot, t) + \int_{\Omega_g \times V} (|g_\epsilon - G_\epsilon| \psi_2)(\cdot, \cdot, t) \\
 & + \int_{\Omega_l \times V \times (0, t)} \frac{1 + \alpha}{\epsilon_l} \psi_1 |f_\epsilon - F_\epsilon| \\
 & + \int_{\Gamma_1^+ \times (0, t)} a(v) \cdot n |f_\epsilon - F_\epsilon| \psi_1 + \int_{\Gamma_2^+ \times (0, t)} a(v) \cdot n_g |g_\epsilon - G_\epsilon| \psi_2 \\
 (21) \quad & + \int_{\Omega_g \times V \times (0, t)} \frac{\alpha}{\epsilon_l} \psi_2 |g_\epsilon - G_\epsilon| \leq \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, t)} |f_\epsilon^n - f_\epsilon^m| \psi_1.
 \end{aligned}$$

Hence we obtain

$$(22) \quad \int_{\Omega_l \times V \times (0, t)} \psi_1 |f_\epsilon^{n+1} - f_\epsilon^{m+1}| \leq \frac{1}{1 + \alpha} \int_{\Omega_l \times V \times (0, t)} |f_\epsilon^n - f_\epsilon^m| \psi_1.$$

This and a reuse of (21) proves that the iterations are contracted to a fixed point in $L^\infty((0, T); L^1(\Omega_l \times V)) \times L^\infty((0, T); L^1(\Omega_g \times V))$, which satisfies Eqs. (1) and (4) and also the boundary and initial conditions (2)–(3) and (5)–(7) (use the arbitrariness of the functions ψ_1 and ψ_2 together with the above convergence). We also infer from the inequality (15) that the solution (f_ϵ, g_ϵ) depends continuously on the initial and boundary data. The fact that (f_ϵ, g_ϵ) satisfies the integral representation is a direct consequence of the characteristic method. The proof of Theorem 2.1 is now finished. \square

Remark 2.1. The proof of the well-posedness of the uncoupled kinetic equations relies on a fixed method based on the integral representation of the solution (see [16, 26]). Because of the transmission boundary conditions, a direct application of the integral representation fails to give contracting operators. It is possible, however, to prove the continuous dependence of the solution on the data using an alternative form of the integral representation.

2.2. Kinetic entropy. We shall prove the following entropy inequality for the solution of the kinetic system.

Theorem 2.2. *The solution to the kinetic system satisfies the relations*

$$\begin{aligned}
 (23) \quad & - \int_{\Omega_l \times V \times (0, T)} (\partial_t + a(v) \cdot \partial_x)(\psi_1) |f_\epsilon - \chi_k| + \int_{\Gamma_i^- \times (0, T)} a(v) \cdot n \psi_1 |g_\epsilon - \chi_k| \\
 & + \int_{\Gamma_1^- \times (0, T)} a(v) \cdot n \psi_1 |f_{\epsilon 0} - \chi_k| \\
 & = - \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n \psi_1 |f_\epsilon - \chi_k| - \int_{\Gamma_1^+ \times (0, T)} a(v) \cdot n \psi_1 |f_\epsilon - \chi_k| \\
 & + \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, T)} (\chi_{u_\epsilon} - f_\epsilon) \psi_1 \operatorname{sgn}(f_\epsilon - \chi_k)
 \end{aligned}$$

and

$$\begin{aligned}
(24) \quad & - \int_{\Omega_g \times V \times (0,T)} (\partial_t + a(v) \cdot \partial_x)(\psi_2)|g_\epsilon - \chi_k| + \int_{\Gamma_1^+ \times (0,T)} a(v) \cdot n_g \psi_2 |f_\epsilon - \chi_k| \\
& + \int_{\Gamma_2^- \times (0,T)} a(v) \cdot n_g \psi_2 |g_{\epsilon 1} - \chi_k| \leq 0 \\
& \forall \psi_1 \in C_0^1(\bar{\Omega}_l \times V \times (0,T)), \forall \psi_2 \in C_0^1(\bar{\Omega}_g \times (0,T)), \psi_2 \geq 0, \forall k \in \mathbb{R}.
\end{aligned}$$

Proof. Multiplying Eq. (1) by $\varphi_1 = \text{sgn}^\mu(f_\epsilon - \chi_k)\psi_1(x, t)$ and Eq. (4) by $\varphi_2 = \text{sgn}^\mu(g_\epsilon - \chi_k)\psi_2(x, t)$ (see the proof of Theorem 2.1) and proceeding as in the proof of Theorem 2.1, we obtain Eq. (23) and similar equation for g_ϵ . Now using the properties of χ , we obtain Eq. (24). The theorem is now proved. \square

2.3. Basic estimates of the solution. We shall state and prove here some basic estimates for the solution of the kinetic system. We begin with L^∞ estimates of f_ϵ and g_ϵ .

Lemma 2.1. *Assume that*

$$\begin{aligned}
& \|f_{\epsilon 0}\|_{L^\infty(\Gamma_1^- \times [0,T])} < C_1, \quad \|g_\epsilon^0\|_{L^\infty(\Omega_g \times V)} < C_2, \\
& \|f_\epsilon^0\|_{L^\infty(\Omega_l \times V)} < C_3, \quad \|g_{\epsilon 1}\|_{L^\infty(\Gamma_2^- \times [0,T])} < C_4
\end{aligned}$$

with C_1, C_2, C_3 , and C_4 independent of ϵ . Then f_ϵ and g_ϵ are uniformly bounded resp. in $L^\infty(\Omega_l \times V \times [0, T])$ and $L^\infty(\Omega_g \times V \times [0, T])$. Moreover we have

$$\begin{aligned}
\|f_\epsilon\|_\infty + \|g_\epsilon\|_\infty & \leq 2\|f_{\epsilon 0}\|_{L^\infty(\Gamma_1^- \times [0,T])} + 2\|g_\epsilon^0\|_{L^\infty(\Omega_g \times V)} + 2\|f_\epsilon^0\|_{L^\infty(\Omega_l \times V)} \\
& + 2\|g_{\epsilon 1}\|_{L^\infty(\Gamma_2^- \times [0,T])} + 8.
\end{aligned}$$

Finally, we have that $f_\epsilon|_{\partial\Omega_l^+ \times (0,T)}$ is uniformly bounded in $L^\infty(\partial\Omega_l^+ \times [0, T])$ and $g_\epsilon|_{\partial\Omega_g^+ \times (0,T)}$ is uniformly bounded in $L^\infty(\partial\Omega_g^+ \times [0, T])$.

The proof of this lemma is a direct consequence of the integral representation of the solution given in Theorem 2.1(i) and the assumptions on the initial and boundary data; see also [25].

Next we present estimates of f_ϵ , g_ϵ , u_ϵ and w_ϵ in $L^\infty([0, T]; L^1(\Omega_l \times V))$, $L^\infty([0, T]; L^1(\Omega_g \times V))$, $L^\infty([0, T]; L^1(\Omega_l))$ and $L^\infty([0, T]; L^1(\Omega_g))$, respectively.

Lemma 2.2. *Assume that*

$$\begin{aligned}
& \|f_\epsilon^0\|_{L^1(\Omega_l \times V)} < C_1, \quad \|g_\epsilon^0\|_{L^1(\Omega_g \times V)} < C_2, \\
& \|a(v) \cdot n_l f_{\epsilon 0}\|_{L^1(\Gamma_1^- \times (0,T))} < C_3, \quad \|a(v) \cdot n_g g_{\epsilon 1}\|_{L^1(\Gamma_2^- \times (0,T))} < C_4
\end{aligned}$$

with C_1, C_2, C_3 , and C_4 positive constant independent of ϵ . Then f_ϵ and g_ϵ are uniformly bounded resp. in $L^\infty([0, T]; L^1(\Omega_l \times V))$ and $L^\infty([0, T]; L^1(\Omega_g \times V))$, and u_ϵ and w_ϵ are uniformly bounded resp. in $L^\infty([0, T]; L^1(\Omega_l))$ and $L^\infty([0, T]; L^1(\Omega_g))$. Moreover, we have

$$\begin{aligned}
& \|u_\epsilon\|_{L^\infty([0,T]; L^1(\Omega_l))} + \|w_\epsilon\|_{L^\infty([0,T]; L^1(\Omega_g))} \\
& \leq \|f_\epsilon\|_{L^\infty([0,T]; L^1(\Omega_l \times V))} + \|g_\epsilon\|_{L^\infty([0,T]; L^1(\Omega_g \times V))} \\
& \leq \|a(v) \cdot n_l f_{\epsilon 0}\|_{L^1(\Gamma_1^- \times (0,T))} + \|a(v) \cdot n_g g_{\epsilon 1}\|_{L^1(\Gamma_2^- \times (0,T))} \\
& + \|f_\epsilon^0\|_{L^1(\Omega_l \times V)} + \|g_\epsilon^0\|_{L^1(\Omega_g \times V)}.
\end{aligned}$$

The proof is a direct consequence of formula (15) with $F_\epsilon \equiv 0$ and $G_\epsilon \equiv 0$.

Next we shall show that under the conditions that the supports in $v \in V$ of the data are compact, the supports in $v \in V$ of the solutions f_ϵ and g_ϵ remain compactly supported with supports included in a fixed compact set independent of ϵ . We shall also give some information about the speed of propagation $a(v)$.

Lemma 2.3. *Assume that*

$$\begin{aligned} f_\epsilon^0 &\in L^1(\Omega_l \times V), \quad g_\epsilon^0 \in L^1(\Omega_g \times V), \\ a(v) \cdot n_l f_{\epsilon 0} &\in L^1(\Gamma_1^- \times (0, T)), \quad a(v) \cdot n_g g_{\epsilon 1} \in L^1(\Gamma_2^- \times (0, T)) \end{aligned}$$

and

$$\begin{aligned} \|f_{\epsilon 0}\|_{L^\infty(\Gamma_1^- \times [0, T])} &< C_1, \quad \|g_\epsilon^0\|_{L^\infty(\Omega_g \times V)} < C_2, \\ \|f_\epsilon^0\|_{L^\infty(\Omega_l \times V)} &< C_3, \quad \|g_{\epsilon 1}\|_{L^\infty(\Gamma_2^- \times [0, T])} < C_4 \end{aligned}$$

with C_1, C_2, C_3 , and C_4 positive constant independent of ϵ . Assume also that the initial and boundary data f_ϵ^0 , $f_{\epsilon 0}$, g_ϵ^0 , and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of ϵ . Then we have

- (i) u_ϵ and w_ϵ are uniformly bounded resp. in $L^\infty(\Omega_l \times [0, T])$ and $L^\infty(\Omega_g \times [0, T])$.
- (ii) f_ϵ and g_ϵ remain compactly supported in $v \in V$ with supports included in a fixed compact set independent of ϵ .
- (iii) The speed of propagation $a(v)$ is finite.

Proof. (i) Use the integral representation of the solution given in Theorem 2.1 and follow the same lines of the proof of a similar result in the uncoupled case given in [26]. The main difference is the treatment of the contributions to the integral over $\Omega_l \times (0, T)$ and $\Omega_g \times (0, T)$ involving the interface conditions. This can be handled using the integral representations given in Theorem 2.1(i). We skip the details.

(ii) Now set $u_\infty = \sup_{\epsilon > 0} \|u_\epsilon\|_{L^\infty(\Omega_l \times [0, T])}$ and $w_\infty = \sup_{\epsilon > 0} \|w_\epsilon\|_{L^\infty(\Omega_g \times [0, T])}$. The terms χ_{u_ϵ} and χ_{w_ϵ} in the integral representation in Theorem 2.1 are supported by $v \in [-u_\infty, u_\infty] \cup [-w_\infty, w_\infty]$, the other terms are supported by v in the compact supports of the boundary and initial data. Thus, for all $t \in [0, T]$, f_ϵ and g_ϵ remain compactly supported, with compact supports included in $\text{Supp}_v f_\epsilon^0 \cup \text{Supp}_v f_{\epsilon 0} \cup [-u_\infty, u_\infty]$ resp. $\text{Supp}_v g_\epsilon^0 \cup \text{Supp}_v g_{\epsilon 1} \cup [-w_\infty, w_\infty]$, which in turn are included in a fixed compact set independent of ϵ .

(iii) Set $u_\infty = \sup_{\epsilon > 0} \|u_\epsilon\|_{L^\infty(\Omega_l \times [0, T])}$ and $w_\infty = \sup_{\epsilon > 0} \|w_\epsilon\|_{L^\infty(\Omega_g \times [0, T])}$. By (i) above, these definitions make sense. Let $S = \text{supp}_v f_\epsilon^0 \cup \text{supp}_v f_{\epsilon 0} \cup [-u_\infty, u_\infty] \cup \text{supp}_v g_\epsilon^0 \cup \text{supp}_v g_{\epsilon 1} \cup [-w_\infty, w_\infty]$. Then the support in v of f_ϵ resp. g_ϵ is included in S . Now set $a_\infty = \sup_{1 \leq i \leq N, v \in S} |a_i(v)|$. We conclude that $\sup_{1 \leq i \leq N, v \in S_{11} \cup S_{12}} |a_i(v)| \leq a_\infty$, where $S_{11} = \{v \in \text{supp}_v f_\epsilon(x, \cdot, t), (x, t) \in \Omega_l \times (0, T)\}$ and $S_{12} = \{v \in \text{supp}_v g_\epsilon(x, \cdot, t), (x, t) \in \Omega_g \times (0, T)\}$. The lemma is proved. \square

2.4. Gauss-Green formula. Let X be an open set of \mathbb{R}^d . Assume that

$$\Gamma_0 = \{(x, v) \in \partial X \times V : a(v) \cdot n(x) = 0\}$$

has measure 0 in $\partial X \times V$ for the measure $d\gamma dv$. Recall from the introduction that this condition is satisfied for Γ_{0l} and Γ_{0g} .

Let W^p denote the space

$$W^p = \{u \in L^p(X \times V \times (0, T)); (\partial_t + a(v) \cdot \partial_x)u \in L^p(X \times V \times (0, T))\},$$

where $1 \leq p < \infty$. Then we have the following theorem.

Theorem 2.3. *Let K be a compact set of $(\partial X)^- \times (0, T)$ (resp. $(\partial X)^+ \times (0, T)$). Then the mapping*

$$u \rightarrow u|_K,$$

defined on $\mathcal{D}(\bar{X} \times V \times (0, T))$, can be extended to a continuous mapping from W^p to $L^p(K)$. As a consequence the functions of W^p ($p \in [1, +\infty)$) have a trace in $L^p_{loc}((\partial X)^- \times (0, T))$ (resp. $L^p_{loc}((\partial X)^+ \times (0, T))$). Here $(\partial X)^-$ and $(\partial X)^+$ are defined according to the definition of Γ^- and Γ^+ given in the Introduction.

When $a(v) = v$, this result has been proven in [6]. The proof for the case of general $a(v)$ satisfying the assumptions made at the beginning of this paper and above, is based on a slight modification of the proof for the case $a(v) = v$. As a consequence of the above theorem, we have the Green's formula

Theorem 2.4. *Let $f \in W^1$. Then $\forall \psi \in C^1_0(\bar{X} \times V \times (0, T))$ we have*

$$(25) = \int_{X \times V \times (0, T)} [\partial_t + a(v) \cdot \partial_x] f \psi + \int_{X \times V \times (0, T)} f [\partial_t + a(v) \cdot \partial_x] \psi \\ + \int_{(\partial X)^+ \times (0, T)} a(v) \cdot n(x) f \psi d\gamma dv dt + \int_{(\partial X)^- \times (0, T)} a(v) \cdot n(x) f \psi d\gamma dv dt.$$

Now let $U \subset \mathbb{R}^N$ be an open bounded subset whose boundary ∂U is regular Lipschitz deformable [4]. That is, we have

(i) $\forall u \in \partial U, \exists r > 0$ and a Lipschitz map $\gamma : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ such that after rotating and relabelling coordinates if necessary,

$$U \cap Q(u, r) = \{y \in \mathbb{R}^N : \gamma(y_1, \dots, y_{N-1}) < y_N\} \cap Q(u, r),$$

where $Q(u, r) = \{y \in \mathbb{R}^N : |u_i - y_i| \leq r, i = 1, \dots, N\}$.

(ii) $\exists \Psi : \partial U \times [0, 1] \rightarrow \bar{U}$ such that Ψ is a homeomorphism bi-Lipschitz over its image and $\Psi(u, 0) = u, \forall u \in \partial U$.

(iii) Let $\partial U_z \equiv \Psi(\partial U \times \{z\})$, $z \in [0, 1]$ and let U_z be the open subset of U whose boundary is ∂U_z and Ψ is as in (ii). Let $\Psi_z : \partial U \rightarrow \bar{U}$ be such that $\Psi_z(u) = \Psi(u, z)$ for any $(u, z) \in \partial U \times [0, 1]$. Ψ_z and γ satisfy

$$\lim_{z \rightarrow 0+} D\Psi_z \circ \gamma = D\gamma \quad \text{in } L^1_{loc}(\mathcal{O}),$$

where \mathcal{O} denotes the greatest open set such that $\gamma(\mathcal{O}) \subset \partial U$.

When ∂U is such that (i) and (ii) are satisfied, we say that ∂U is Lipschitz deformable and Ψ is a Lipschitz deformation of ∂U . If a Lipschitz deformable boundary ∂U satisfies (iii), then the Lipschitz deformation Ψ of ∂U is called regular. Examples of domains whose boundaries are regular Lipschitz deformable include bounded domains with smooth (C^2 for example) boundaries; star-shaped domains and those satisfying the cone property; see [4].

Let $\nu : \Psi(\partial U \times [0, 1]) \rightarrow \mathbb{R}^N$ be such that $\nu(u)$ is the unit outward normal to ∂U_z at $u \in \partial U_z \equiv \Psi(\partial U \times \{z\})$ defined a.e. We have [4]

Theorem 2.5. *Let F be a function in $L^\infty(U; \mathbb{R}^N)$ whose divergence is a bounded measure. Then there exists a continuous linear functional $F \cdot \nu|_{\partial U}$ over $Lip(\partial U)$ such that*

$$\langle F \cdot \nu|_{\partial U}, \phi \rangle = \int_U \phi \operatorname{div} F + \int_U \nabla \phi \cdot F \quad \forall \phi \in Lip(\mathbb{R}^N).$$

Moreover, the normal trace $F \cdot \nu|_{\partial U}$ is a function in $L^\infty(\partial U)$ satisfying

$$\|F \cdot \nu\|_{L^\infty(\partial U)} \leq C \|F\|_{L^\infty(U)}$$

for some constant C independent of F and

$$\langle F \cdot \nu|_{\partial U}, \psi \rangle = \text{ess limit}_{z \rightarrow 0} \int_{\partial U_z} \psi \circ \Psi_z^{-1} F \cdot \nu d\mathcal{H}^{N-1} \quad \forall \psi \in Lip(\partial U).$$

3. KINETIC FORMULATION OF THE COUPLED KINETIC-HYDRODYNAMIC SYSTEM

We begin by introducing a definition for a physically correct solution to problem (8)-(13).

Definition 3.1. Let $f \in L^\infty([0, T]; L^1(\Omega_l \times V)) \cap L^\infty(\Omega_l \times [0, T]; L^1(V))$ and $w \in L^\infty(\Omega_g \times (0, T)) \cap L^\infty([0, T]; L^1(\Omega_g))$. We say that (f, w) is a weak entropy solution of system (8)-(13) if the equations (8) and (11) are satisfied in the distribution sense, the weak traces of f on $\Gamma_1^+ \times (0, T)$ and $\Gamma_i^+ \times (0, T)$ and χ_w on $\Gamma_i^- \times (0, T)$ exist, and the following holds:

$$\begin{aligned} & - \int_{\Omega_l \times V \times (0, T)} [\partial_t + a(v) \cdot \partial_x](\psi_1) f + \int_{\Gamma_1^- \times (0, T)} a(v) \cdot n_l \psi_1 f_0 \\ & + \int_{\Gamma_1^+ \times (0, T)} a(v) \cdot n_l \psi_1 f + \int_{\Gamma_i^- \times (0, T)} a(v) \cdot n_l \psi_1 \chi_w + \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n_l \psi_1 f \\ (26) \quad & = \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, T)} (\chi_{u(x, t)}(v) - f(x, v, t))(\psi_1) \end{aligned}$$

and

$$\begin{aligned} & - \int_{\Omega_g \times (0, T)} (|w - k| \partial_t \psi_2 + \text{sign}(w - k)(A(w) - A(k)) \cdot \nabla_x \psi_2) \\ & + \int_{\Gamma_2 \times (0, T)} \psi_2 \text{sign}(w_1 - k)((A(w_1) \cdot n_g)^- - (A(k) \cdot n)^-) \\ (27) \quad & + \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n_g \psi_2 |f - \chi_k| \leq 0 \\ & \forall \psi_1 \in C_0^1(\bar{\Omega}_l \times V \times (0, T)), \quad \forall \psi_2 \in C_0^1(\bar{\Omega}_g \times (0, T)), \quad \psi_2 \geq 0, \quad \forall k \in \mathbb{R} \end{aligned}$$

and the initial conditions

$$(28) \quad f(x, v, 0) = f^0(x, v) \text{ in } \Omega_l \times V \quad w(x, 0) = w^0(x) \text{ in } \Omega_g.$$

Remark 3.1. (i) Since by definition $f \in L^\infty([0, T]; L^1(\Omega_l \times V)) \cap L^\infty(\Omega_l \times [0, T]; L^1(V))$, $f \in L^1(\Omega_l \times (0, T) \times L^1(V))$ and $(\partial_t + a(v) \cdot \partial_x)f \in L^1(\Omega_l \times V \times (0, T))$ (see the equation satisfied by f). Thus, we can apply Theorems 2.3 and 2.4. Therefore, the trace of f on $(\partial\Omega_l)^+ \times (0, T)$ is in $L_{loc}^1((\partial\Omega_l)^+ \times (0, T))$. This justifies the integrals involving the boundary terms of f . The existence of the weak trace of χ_w is proved in Theorem 4.10 which is given at the end of the paper. For more details consult the proof of Theorem 4.8.

(ii) Below we shall make precise the boundary conditions for w on $\partial\Omega_g \times (0, T)$. Taking $\psi_2 \in C_0^1(\bar{\Omega}_g \times (0, T))$ in the definition above, we obtain

$$(29) \quad \partial_t |w - k| + \text{div}[\text{sgn}(w - k)(A(w) - A(k))] \leq 0 \text{ in } \mathcal{D}'(\Omega_g \times (0, T)).$$

Therefore $\partial_t |w - k| + \text{div}[\text{sgn}(w - k)(A(w) - A(k))]$ is a bounded measure on $\Omega_g \times (0, T)$. Set $F = (|w - k|, \text{sgn}(w - k)(A(w) - A(k)))$; then it is clear that $F \in L^\infty(\Omega_g \times (0, T); \mathbb{R}^{d+1})$. On the other hand, using the above,

$$\text{div}_{t,x} F = \partial_t |w - k| + \text{div}[\text{sgn}(w - k)(A(w) - A(k))]$$

is a bounded measure on $U = \Omega_g \times (0, T)$. Hence we can use Theorem 2.5, which yields

$$\begin{aligned}
 & \text{ess limit}_{z \rightarrow 0} \int_{\partial U_z} \phi \circ \Psi_z^{-1} F \cdot \nu d\mathcal{H}^d \\
 &= \int_{\Omega_g \times (0, T)} \phi(\partial_t |w - k| + \operatorname{div}[\operatorname{sgn}(w - k)(A(w) - A(k))]) \\
 (30) \quad &+ \int_{\Omega_g \times (0, T)} (|w - k| \partial_t \phi + \operatorname{sgn}(w - k)(A(w) - A(k)) \cdot \nabla_x \phi) \quad \forall \phi \in Lip(\mathbb{R}^{d+1}),
 \end{aligned}$$

where $\nu = (\nu_1, \nu_\star)$ and Ψ were introduced in the previous section. Taking $\psi_2 = \phi \in C_0^1(\bar{\Omega}_g \times (0, T))$, $\psi_2 \geq 0$, in Definition 3.1 and using (29) and (30), we obtain

$$\begin{aligned}
 & -\text{ess limit}_{z \rightarrow 0} \int_{\partial \Omega_g \times (0, T)} \phi \circ \Psi_z^{-1} \operatorname{sgn}(w - k)(A(w) - A(k)) \cdot \nu_\star d\mathcal{H}^d \\
 &+ \int_{\Omega_g \times (0, T)} \phi(\partial_t |w - k| + \operatorname{div}[\operatorname{sgn}(w - k)(A(w) - A(k))]) \\
 &+ \int_{\Gamma_2 \times (0, T)} \psi_2 \operatorname{sign}(w_1 - k)((A(w_1) \cdot n_g)^- - (A(k) \cdot n_g)^-) \\
 &+ \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n_g \psi_2 |f - \chi_k| \leq 0.
 \end{aligned}$$

For $\psi_2 \in C_0^1(\partial \Omega_g \times (0, T))$, this yields

$$\begin{aligned}
 & -\text{ess limit}_{z \rightarrow 0} \int_{\partial \Omega_g \times (0, T)} \phi \circ \Psi_z^{-1} \operatorname{sgn}(w - k)(A(w) - A(k)) \cdot \nu_\star d\mathcal{H}^d \\
 &+ \int_{\Gamma_2 \times (0, T)} \psi_2 \operatorname{sign}(w_1 - k)((A(w_1) \cdot n_g)^- - (A(k) \cdot n_g)^-) \\
 (31) \quad &+ \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n_g \psi_2 |f - \chi_k| \leq 0.
 \end{aligned}$$

This in turn yields

$$\begin{aligned}
 & -\text{ess limit}_{z \rightarrow 0} \int_{\Psi(\Gamma_2 \times \{z\}) \times (0, T)} \psi_2 \circ \Psi_z^{-1} \operatorname{sgn}(w - k)(A(w) - A(k)) \cdot \nu_\star d\mathcal{H}^d \\
 &+ \int_{\Gamma_2 \times (0, T)} \psi_2 \operatorname{sign}(w_1 - k)((A(w_1) \cdot n_g)^- - (A(k) \cdot n_g)^-) \leq 0 \\
 (32) \quad & \forall \psi_2 \in C_0^1(\Gamma_2 \times (0, T))
 \end{aligned}$$

and

$$\begin{aligned}
 & -\text{ess limit}_{z \rightarrow 0} \int_{\Psi(\Gamma_i \times \{z\}) \times (0, T)} \psi_2 \circ \Psi_z^{-1} \operatorname{sgn}(w - k)(A(w) - A(k)) \cdot \nu_\star d\mathcal{H}^d \\
 &+ \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n_g \psi_2 |f - \chi_k| \leq 0 \\
 (33) \quad & \forall \psi_2 \in C_0^1(\Gamma_i \times (0, T)).
 \end{aligned}$$

This clarifies the boundary conditions for w .

Let $f_l(x, v, t)$ and $g(x, v, t)$ be functions defined respectively in $\Omega_l \times V \times (0, T)$ and $\Omega_g \times V \times (0, T)$. We introduce the following system:

$$(34) \quad [\partial_t + a(v) \cdot \partial_x] f_l(x, v, t) = \frac{1}{\epsilon_l} (\chi_{u_l(x, t)}(v) - f_l(x, v, t)) \quad \text{in } \Omega_l \times V \times (0, T)$$

$$(35) \quad f_l(x, v, t) = f_0(x, v, t) \quad \text{on } \Gamma_1^- \times (0, T),$$

$$(36) \quad f_l(x, v, t) = g(x, v, t) \quad \text{on } \Gamma_i^- \times (0, T)$$

$$(37) \quad [\partial_t + a(v) \cdot \partial_x] g(x, v, t) = \frac{\partial m}{\partial v} \quad \text{in } \Omega_g \times V \times (0, T)$$

$$(38) \quad g(x, v, t) = \chi_{w(x, t)}(v) \quad \text{in } \Omega_g \times V \times (0, T),$$

$$(39) \quad g(x, v, t) = \chi_{w_1(x, t)}(v) \quad \text{on } \Gamma_2^- \times (0, T),$$

$$(40) \quad g(x, v, t) = f_l(x, v, t) \quad \text{on } \Gamma_i^+ \times (0, T),$$

$$(41) \quad m \text{ nonnegative bounded measure in } \Omega_g \times V \times (0, T),$$

$$(42) \quad f_l(x, v, 0) = f^0(x, v) \quad \text{in } \Omega_l \times V \quad g(x, v, 0) = g^0(x, v) \quad \text{in } \Omega_g \times V,$$

where f_0 , w_1 , and f^0 were introduced in Section 1. The function u_l is related to f_l by $u_l = \int_V f_l$. The function g^0 is related to w^0 by

$$(43) \quad g^0(x, v) = \chi_{w^0(x)}(v) \quad \text{in } \Omega_g \times V.$$

The boundary conditions in (35), (36), (39), and (40) will be made precise in the following definition of a solution to the above system.

Definition 3.2. We say that

$$(f_l, g) \in (L^\infty([0, T]; L^1(\Omega_l \times V)) \cap L^\infty(\Omega_l \times [0, T]; L^1(V))) \\ \times (L^\infty([0, T]; L^1(\Omega_g \times V)) \cap L^\infty(\Omega_g \times (0, T); L^1(V)))$$

is a weak solution of the system (34)-(42) if the equations (34) and (37) are satisfied in the distribution sense, the weak traces of f_l on $\Gamma_1^+ \times (0, T)$ and on $\Gamma_i^+ \times (0, T)$ and g on $\Gamma_i^- \times (0, T)$ exist, and we have

$$(44) \quad - \int_{\Omega_l \times V \times (0, T)} (\partial_t + a(v) \cdot \partial_x)(\psi_1) f_l + \int_{\Gamma_1^- \times (0, T)} a(v) \cdot n \psi_1 f_0 \\ + \int_{\Gamma_1^+ \times (0, T)} a(v) \cdot n \psi_1 f_l + \int_{\Gamma_i^- \times (0, T)} a(v) \cdot n \psi_1 g \\ + \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n \psi_1 f_l = \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, T)} (\chi_{u_l(x, t)}(v) - f_l(x, v, t))(\psi_1)$$

and

$$(45) \quad - \int_{\Omega_g \times V \times (0, T)} (\partial_t + a(v) \cdot \partial_x)(\psi_2) |g - \chi_k| + \int_{\Gamma_2^- \times (0, T)} a(v) \cdot n \psi_2 |\chi_{w_1} - \chi_k| \\ + \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n \psi_2 |f_l - \chi_k| \leq 0$$

$$\forall \psi_1 \in C_0^1(\bar{\Omega}_l \times V \times (0, T)), \quad \forall \psi_2 \in C_0^1(\bar{\Omega}_g \times (0, T)), \quad \psi_2 \geq 0, \quad \forall k \in \mathbb{R}$$

and the initial conditions

$$(46) \quad f_l(x, v, 0) = f^0(x, v) \quad \text{in } \Omega_l \times V \quad g(x, v, 0) = g^0(x, v) \quad \text{in } \Omega_g \times V.$$

Remark 3.2. Notice that Remark 3.1 applies here also.

The following theorem states the equivalence between the two systems (8)-(13) and (34)-(42).

Theorem 3.1. (i) Let $f \in L^\infty([0, T]; L^1(\Omega_l \times V)) \cap L^\infty(\Omega_l \times [0, T]; L^1(V))$ and $w \in L^\infty(\Omega_g \times [0, T]) \cap L^\infty([0, T]; L^1(\Omega_g))$ solve (8)-(13) in the sense of Definition 3.1. Set $f_l = f$ and $g(x, v, t) = \chi_{w(x, t)}(v)$. Then $f_l \in L^\infty([0, T]; L^1(\Omega_l \times V)) \cap L^\infty(\Omega_l \times [0, T]; L^1(V))$ and $g \in L^\infty([0, T]; L^1(\Omega_g \times V)) \cap L^\infty(\Omega_g \times [0, T]; L^1(V))$ solve in the sense of Definition 3.2 the system (34)-(42) for some m supported in $\Omega_g \times [-R_g, R_g] \times (0, T)$ with $R_g = \|w\|_{L^\infty}$.

(ii) Let $f_l \in L^\infty([0, T]; L^1(\Omega_l \times V)) \cap L^\infty(\Omega_l \times [0, T]; L^1(V))$ and $g \in L^\infty([0, T]; L^1(\Omega_g \times V)) \cap L^\infty(\Omega_g \times [0, T]; L^1(V))$ solve (34)-(42). Set $w(x, t) = \int_V g(x, v, t) dv$ and $f = f_l$. Then $w \in L^\infty(\Omega_g \times [0, T]) \cap L^\infty([0, T]; L^1(\Omega_g))$ and (f, w) solves (8)-(13) in the sense of Definition 3.1.

Remark 3.3. The kinetic formulation of the coupled system (8)-(13) given above is inspired from the kinetic formulation of conservation laws with boundaries introduced by the author in [27], which is in turn based on the generalization of kinetic formulation introduced in [12] to conservation laws for domains with boundaries.

Proof of Theorem 3.1. We begin with the proof of (i). The construction of m , the proof of its boundedness, and the proof that g and m satisfy Eq. (37) are similar to those given in [12]. Therefore we refer to that paper and give detail only for the new aspects of the formulation, which are related to the boundary and transmission conditions.

Using Eq. (26) in Definition 3.1 we conclude that (f_l, g) satisfies the first part of Definition 3.2, i.e. Eq. (44). Now using Eq. (27) and the properties of χ , we conclude that (f_l, g) satisfies the second part of Definition 3.2, i.e. Eq. (45). Finally, since (f, w) satisfies the initial condition (28), using (43) and the properties of χ , (f_l, w) satisfies the initial condition (46).

We now prove the second part of the theorem. Eq. (8) is clearly satisfied. On the other hand

$$(47) \quad \left\langle \frac{\partial g}{\partial t} + \sum_{i=1}^d a_i(v) \cdot \frac{\partial g}{\partial x_i}, h \otimes 1 \right\rangle = 0$$

$\forall h \in \mathcal{D}(\Omega_g \times (0, T))$. Using the properties of χ , we conclude that Eq. (11) is satisfied in the distribution sense.

We now wish to prove that the remaining requirements in Definition 3.1 are satisfied. First, using Eq. (44), Eq. (26) is satisfied. To prove that Eq. (27) is satisfied, we just use Eq. (45) and the properties of χ . Finally, since (f_l, g) satisfies the initial condition (46), using (43) and the properties of χ , (f, w) satisfies the initial condition (28). The theorem is now proved. \square

4. ANALYSIS OF THE KINETIC FORMULATION

In this section we shall study the behavior of system (1)-(7), as the microscopic scale in Ω_g tends to 0. We prove that as $\epsilon \rightarrow 0$, the solution of the coupled kinetic system converges to a solution of system (34)-(42). The existence theory for the system (8)-(13) will be deduced from Theorem 3.1 using the kinetic formulation of the coupled system.

In the sequel, we shall assume that $\Omega_l = (0, l)$ and $\Omega_g = (l, 1)$. It is not difficult to generalize our proof to the case $\Omega_l = (0, l) \times \mathbb{R}^{d-1}$ and $\Omega_g = (l, 1) \times \mathbb{R}^{d-1}$. Throughout the rest of this paper the terminology convergence of h_ϵ as $\epsilon \rightarrow 0$ means convergence up to a subsequence of h_ϵ , still denoted h_ϵ , and does not necessarily mean convergence of the whole sequence. Also $\lim_{z \rightarrow 0}$ will simply be denoted by $\lim_{z \rightarrow 0}$.

We first introduce a nondegeneracy assumption on the curve: $v \rightarrow a(v)$ in \mathbb{R}^d . This relation is needed in order to use regularity results for the averages. Assume that the initial and boundary data satisfy

$$\begin{aligned} \|f_\epsilon^0\|_{L^1(\Omega_l \times V)} &< C_1, \quad \|g_\epsilon^0\|_{L^1(\Omega_g \times V)} < C_2, \\ \|a(v) \cdot n_l f_{\epsilon 0}\|_{L^1(\Gamma_1^- \times (0, T))} &< C_3, \quad \|a(v) \cdot n_g g_{\epsilon 1}\|_{L^1(\Gamma_2^- \times (0, T))} < C_4, \\ \|f_{\epsilon 0}\|_{L^\infty(\Gamma_1^- \times [0, T])} &< C_5, \quad \|g_\epsilon^0\|_{L^\infty(\Omega_g \times V)} < C_6, \\ \|f_\epsilon^0\|_{L^\infty(\Omega_l \times V)} &< C_7, \quad \|g_{\epsilon 1}\|_{L^\infty(\Gamma_2^- \times [0, T])} < C_8 \end{aligned}$$

with C_1, \dots, C_8 positive constants independent of ϵ . Assume also that the initial and boundary data f_ϵ^0 , $f_{\epsilon 0}$, g_ϵ^0 , and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of ϵ . Then by Lemma 2.2 u_ϵ resp. w_ϵ is uniformly bounded in $L^\infty([0, T]; L^1(\Omega_l))$ resp. $L^\infty([0, T]; L^1(\Omega_g))$, and by Lemma 2.3 u_ϵ resp. w_ϵ is uniformly bounded in $L^\infty(\Omega_l \times [0, T])$ resp. in $L^\infty(\Omega_g \times [0, T])$. Let R_∞ denote the common bound of u_ϵ in $L^\infty([0, T]; L^1(\Omega_l))$ and $L^\infty(\Omega_l \times [0, T])$ and of w_ϵ in $L^\infty([0, T]; L^1(\Omega_g))$ and $L^\infty(\Omega_g \times [0, T])$. We then make the following assumption on the curve $a(\cdot)$:

$$(48) \quad \text{meas}\{|v| < R_\infty, \tau + a(v) \cdot \xi = 0\} = 0 \quad \forall (\tau, \xi) \in \mathbb{R} \times \Omega_g \text{ such that } \tau^2 + \xi^2 = 1.$$

This condition was introduced in [12]. In the case $d = 1$ this relation is actually stronger than a related condition introduced by Tartar [18]. For the description of examples of fluxes A satisfying (48) and examples of fluxes A where this condition fails, we refer to [12].

We begin with a compactness result for u_ϵ .

Theorem 4.1. *Assume that*

$$\begin{aligned} \|f_{\epsilon 0}\|_{L^\infty(\Gamma_1^- \times [0, T])} &< C_1, \quad \|g_\epsilon^0\|_{L^\infty(\Omega_g \times V)} < C_2, \quad \|f_\epsilon^0\|_{L^\infty(\Omega_l \times V)} < C_3, \\ \|g_{\epsilon 1}\|_{L^\infty(\Gamma_2^- \times [0, T])} &< C_4, \quad \|f_\epsilon^0\|_{L^1(\Omega_l \times V)} < C_5, \quad \|g_\epsilon^0\|_{L^1(\Omega_g \times V)} < C_6, \\ \|a(v) \cdot n_l f_{\epsilon 0}\|_{L^1(\Gamma_1^- \times (0, T))} &< C_7, \quad \|a(v) \cdot n_g g_{\epsilon 1}\|_{L^1(\Gamma_2^- \times (0, T))} < C_8 \end{aligned}$$

with $C_i, i = 1, \dots, 8$, positive constants independent of ϵ .

Assume also that the initial and boundary data f_ϵ^0 , $f_{\epsilon 0}$, g_ϵ^0 , and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of ϵ . Finally assume that (48) hold. Then a subsequence of $u_\epsilon = \int_{\mathbb{R}} f_\epsilon dv$ (still denoted u_ϵ) converges strongly in $L^1(\Omega_l \times (0, T))$ to a function $u \in L^1(\Omega_l \times (0, T))$.

The proof of this statement is based on the following averaging lemma.

Theorem 4.2. *Let $f \in L^q(\Omega_l \times V \times (0, T))$, let*

$$\partial f / \partial t + a(v) \cdot \nabla_x f \in L^q(\Omega_l \times V \times (0, T)), \quad q > 1,$$

let $f|_{t=0} = f^0 \in L^\infty(\Omega_l \times V)$, and let $f|_{(\partial\Omega_l)^-} = f_b \in L^\infty((\partial\Omega_l)^- \times (0, T))$. Assume also that (48) is satisfied.

Then $\int_V f dv \in W^{s,q}(\Omega_l \times (0, T))$ for any s , $0 < s < \inf(1/q, 1 - 1/q)$, and

$$\left\| \int_V f dv \right\|_{W^{s,q}} \leq C(f_b, f^0) (\|f\|_{L^q} + \|\partial f / \partial t + a(v) \cdot \nabla_x f\|_{L^q}).$$

When $a(v) = v$ Theorem 4.2 is exactly the same as the averaging lemma given in [9, 1] and proved in [9]. Therefore our lemma is a slight generalization of the lemma in [9, 1]. The proof is similar to the proof given in [9].

Proof of Theorem 4.1. First, we observe that Lemmas 2.2-2.3 show that f_ϵ and $\partial f_\epsilon / \partial t + a(v) \cdot \nabla_x f_\epsilon$ are uniformly bounded in $L^q(\Omega \times V \times (0, T))$ for any $q \geq 1$. Lemma 2.1, together with the transmission condition (3) and the integral representation in Theorem 2.1, shows that $f_\epsilon|_{\Gamma_i^-}$ is uniformly bounded in $L^\infty(\Gamma_i^- \times (0, T))$. Moreover, we know by assumption that $f_{\epsilon 0}$ resp. f_ϵ^0 is uniformly bounded in $L^\infty(\Gamma_1^- \times (0, T))$ resp. $L^\infty(\Omega_l \times V)$. Therefore we can use Theorem 4.2 and the uniform boundedness of u_ϵ in $L^\infty(\Omega_l \times (0, T))$ to infer that a subsequence of u_ϵ (still denoted u_ϵ) converges in $L^1(\Omega_l \times (0, T))$ to a function $u \in L^1(\Omega_l \times (0, T))$. This concludes the proof of the theorem. \square

Theorem 4.3. Assume that

$$\begin{aligned} \|f_{\epsilon 0}\|_{L^\infty(\Gamma_1^- \times [0, T])} &< C_1, \quad \|g_\epsilon^0\|_{L^\infty(\Omega_g \times V)} < C_2, \quad \|f_\epsilon^0\|_{L^\infty(\Omega_l \times V)} < C_3, \\ \|g_{\epsilon 1}\|_{L^\infty(\Gamma_2^- \times [0, T])} &< C_4, \quad \|f_\epsilon^0\|_{L^1(\Omega_l \times V)} < C_5, \quad \|g_\epsilon^0\|_{L^1(\Omega_g \times V)} < C_6, \\ \|a(v) \cdot n_l f_{\epsilon 0}\|_{L^1(\Gamma_1^- \times (0, T))} &< C_7, \quad \|a(v) \cdot n_g g_{\epsilon 1}\|_{L^1(\Gamma_2^- \times (0, T))} < C_8 \end{aligned}$$

with $C_i, i = 1, \dots, 8$, positive constants independent of ϵ .

Assume also that the initial and boundary data f_ϵ^0 , $f_{\epsilon 0}$, g_ϵ^0 , and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of ϵ . Finally assume that (48) holds. Then:

- (i) A subsequence of f_ϵ (still denoted f_ϵ) converges in L^∞ weak- \star to a function $f \in L^\infty(\Omega_l \times V \times [0, T])$ and f satisfies Eq. (34).
- (ii) A subsequence of f_ϵ (still denoted f_ϵ) converges strongly in $L^1(\Omega_l \times V \times (0, T))$ to a function $f \in L^1(\Omega_l \times V \times (0, T))$ and $u = \int_V f$.

The proof of the first point in (ii) of Theorem 4.3 relies also on the following result [26].

Theorem 4.4. Let U be a bounded open subset of \mathbb{R}^N and let v_n be a sequence in $L^1_{loc}(U)$. Assume that as $n \rightarrow \infty$, the sequence v_n converges strongly in $L^1_{loc}(U)$ to $v \in L^1_{loc}(U)$. If v_n is uniformly bounded in $L^\infty(U)$, then v_n converges strongly to v in $L^1(U)$.

Proof of Theorem 4.3. (i) Using Theorem 4.1 a subsequence of u_ϵ converges strongly in $L^1(\Omega_l \times (0, T))$ to a function u . Hence by the properties of χ , χ_{u_ϵ} converges strongly to χ_u in $L^1(\Omega_l \times V \times (0, T))$. Using Lemma 2.1 a subsequence of f_ϵ (still denoted f_ϵ) converges, as $\epsilon \rightarrow 0$, in $L^\infty(\Omega_l \times V \times [0, T])$ weak- \star to a function $f \in L^\infty(\Omega_l \times V \times [0, T])$. Moreover, $(\partial_t + a(v) \cdot \partial_x) f_\epsilon$ converges to $(\partial_t + a(v) \cdot \partial_x) f$ in the distribution sense. We also know from above and from the diagonal process to subtract a further subsequence, if necessary, that a subsequence of χ_{u_ϵ} converges as $\epsilon \rightarrow 0$ to χ_u strongly in $L^1(\Omega_l \times V \times (0, T))$ and that $f \in L^\infty(\Omega_l \times V \times [0, T])$. Therefore f satisfies Eq. (34) a.e.

(ii) The function f_ϵ satisfies Eq. (1). Subtracting this equation from Eq. (34), and multiplying the resulting equation by φ , a test function in $C^1(\Omega_l \times V \times [0, T])$, with compact support in x to be made precise later, and integrating by parts, we obtain

$$\begin{aligned}
 & \int_{\Omega_l \times V} ((f_\epsilon - f)\varphi)(\cdot, \cdot, t) - \int_{\Omega_l \times V} ((f_\epsilon - f)\varphi)(\cdot, \cdot, 0) \\
 & - \int_{\Omega_l \times V \times (0, t)} (\partial_t + a(v) \cdot \partial_x)(\varphi)(f_\epsilon - f) \\
 (49) \quad & = \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, t)} ((\chi_{u_\epsilon} - \chi_u) - (f_\epsilon - f))\varphi.
 \end{aligned}$$

We recall that the trace of f at the initial time exists since $(\partial_t + a(v) \cdot \partial_x)f$ and f are in $L^1 \cap L^\infty$. We then take $\varphi = \text{sign}^\mu(f_\epsilon - f)\psi(x, t)$ with $x\text{sign}^\mu(x) \geq 0$, $x \in \mathbb{R}$, where ψ is a nonnegative test function and sign^μ is a regularization of sign function. Plugging in (49) and passing to the limit as $\mu \rightarrow 0$, we obtain

$$\begin{aligned}
 & \int_{\Omega_l \times V} (|f_\epsilon - f|\psi)(\cdot, \cdot, t) + \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, t)} |f_\epsilon - f|\psi \\
 & = \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, t)} (\chi_{u_\epsilon} - \chi_u)\text{sign}(f_\epsilon - f)\psi + \int_{\Omega_l \times V} (|f_\epsilon - f|\psi)(\cdot, \cdot, 0) \\
 (50) \quad & + \int_{\Omega_l \times V \times (0, t)} (\partial_t + a(v) \cdot \partial_x)(\psi)|f_\epsilon - f|.
 \end{aligned}$$

In particular we have

$$\begin{aligned}
 & \int_{O \times V} (|f_\epsilon - f|\psi)(\cdot, \cdot, t) + \frac{1}{\epsilon_l} \int_{O \times V \times (0, t)} |f_\epsilon - f|\psi \\
 (51) \quad & = \frac{1}{\epsilon_l} \int_{O \times V \times (0, t)} (\chi_{u_\epsilon} - \chi_u)\text{sign}(f_\epsilon - f)\psi \\
 & + \int_{O \times V \times (0, t)} (\partial_t + a(v) \cdot \partial_x)(\psi)|f_\epsilon - f|
 \end{aligned}$$

for any open set with $\bar{O} \subset \Omega_l$, and ψ for any Lipschitz continuous function in $O \times V \times [0, T]$ with compact support in x in O and such that $\psi(\cdot, \cdot, 0) \equiv 0$.

Let U and O be open bounded subsets of Ω_l such that $\bar{U} \subset O \subset \bar{O} \subset \Omega_l$. Let ψ be a Lipschitz continuous function in $O \times V \times [0, T]$ with compact support in x in O and satisfying $\psi(\cdot, \cdot, 0) \equiv 0$, such that $U \subset \text{supp}_x \psi \subset O$. Then (52) holds for such ψ and O .

We wish to prove that

$$(52) \quad \int_{U \times V} |f_\epsilon - f|(\cdot, \cdot, t) + \frac{1}{\epsilon_l} \int_{U \times V \times (\eta, t)} |f_\epsilon - f| \xrightarrow{\epsilon \rightarrow 0} 0$$

for any $t \in (0, T]$ and any $0 < \eta < t \leq T$. It is enough to prove this relation for U of the form $U = (y - \alpha, y + \alpha)$, where $\alpha > 0$ is any positive real such that $0 < y - \alpha < y + \alpha < l$. Let $\beta > 0$ and $\gamma > 0$ be such that $0 < y - \alpha - \beta - \gamma < y + \alpha + \beta + \gamma < l$. Let $0 < t_1 < T$ be such that $a_\infty t_1 = \beta$. Let $O = (y - \alpha - a_\infty t_1 - \gamma, y + \alpha + a_\infty t_1 + \gamma)$

and let $t \in (0, t_1]$. Now consider the function

$$\psi_1(x, \tau) = \begin{cases} 0 & 0 \leq x < y - \alpha - a_\infty(t - \tau) - \gamma, \\ & 0 \leq \tau \leq t, \\ \frac{1}{\gamma}(x - y + \alpha + a_\infty(t - \tau)) + 1 & y - \alpha - a_\infty(t - \tau) - \gamma \leq x \\ & < y - \alpha - a_\infty(t - \tau), \quad 0 \leq \tau \leq t, \\ 1 & y - \alpha - a_\infty(t - \tau) \leq x < y + \alpha \\ & + a_\infty(t - \tau), \quad 0 \leq \tau \leq t, \\ \frac{1}{\gamma}(y + \alpha + a_\infty(t - \tau) - x) + 1 & y + \alpha + a_\infty(t - \tau) \leq x < y + \alpha \\ & + a_\infty(t - \tau) + \gamma, \quad 0 \leq \tau \leq t, \\ 0 & y + \alpha + a_\infty(t - \tau) + \gamma \leq x \leq l, \\ & 0 \leq \tau \leq t, \end{cases}$$

and

$$\psi_2(x, \tau) = e^{-\frac{1}{\tau}}$$

Now let $\psi(x, \tau) = \psi_1(x, \tau)\psi_2(x, \tau)$, $\tau \in [0, t]$, and $x \in \Omega_l$. It is clear that ψ is a nonnegative Lipschitz continuous function in $O \times V \times [0, T]$ with compact support in x in O and $U \subset \text{supp}_x \psi \subset O$. Thus, plugging ψ in (52) and using the fact that χ_{u_ϵ} converges strongly in $L^1(\Omega_l \times V \times (0, T))$ to χ_u yields (52) for $t \in (0, t_1]$. Now let $t_2 > t_1$ be such that $a_\infty(t_2 - t_1) = \beta$. Proceeding as above and using the fact that $\int_{U \times V} |f_\epsilon - f|(\cdot, \cdot, t_1)$ converges to 0 as $\epsilon \rightarrow 0$, we conclude that $\int_{U \times V} |f_\epsilon - f|(\cdot, \cdot, t)$ and $\frac{1}{\epsilon l} \int_{U \times V \times (t_1, t_2)} |f_\epsilon - f|$ converge to 0 for any $t \in (t_1, t_2]$. Continuing this process we conclude that $\int_{U \times V} |f_\epsilon - f|(\cdot, \cdot, t)$ and $\frac{1}{\epsilon l} \int_{U \times V \times (\eta, T)} |f_\epsilon - f|$ converge to 0 for any $t \in (0, T]$ and any $0 < \eta < t \leq T$.

Therefore, f_ϵ converges in $L^1_{loc}(\Omega_l \times (0, T); L^1(V)) \cap L^\infty([\eta, T]; L^1_{loc}(\Omega_l; L^1(V)))$ to f . Now using Lemma 2.3 $\text{supp}_v f_\epsilon$ is included into a fixed compact set independent of ϵ . Therefore, since f_ϵ converges to f in L^1_{loc} and f_ϵ is uniformly bounded in $L^\infty(\Omega_l \times V \times [0, T])$ (Lemma 2.1), we may use Theorem 4.4 to infer that f_ϵ converges strongly to f in $L^1(\Omega_l \times V \times (0, T))$.

Finally, since $\int_{\Omega_l \times (0, T)} |u_\epsilon - \int_V f| \leq \int_{\Omega_l \times V \times (0, T)} |f_\epsilon - f|$ and since the limit is unique, $u = \int_V f$. The proof of the theorem is now completed. \square

Remark 4.1. Notice that in the course of proving Theorem 4.3, we have proved that f_ϵ converges strongly to f in $L^\infty([\eta, T]; L^1(\Omega_l \times V))$ for any $0 < \eta < T$.

Theorem 4.5. *Assume that*

$$\begin{aligned} \|f_{\epsilon 0}\|_{L^\infty(\Gamma_1^- \times [0, T])} &< C_1, \quad \|g_\epsilon^0\|_{L^\infty(\Omega_g \times V)} < C_2, \quad \|f_\epsilon^0\|_{L^\infty(\Omega_l \times V)} < C_3, \\ \|g_{\epsilon 1}\|_{L^\infty(\Gamma_2^- \times [0, T])} &< C_4, \quad \|f_\epsilon^0\|_{L^1(\Omega_l \times V)} < C_5, \quad \|g_\epsilon^0\|_{L^1(\Omega_g \times V)} < C_6, \\ \|a(v) \cdot n_l f_{\epsilon 0}\|_{L^1(\Gamma_1^- \times (0, T))} &< C_7, \quad \|a(v) \cdot n_g g_{\epsilon 1}\|_{L^1(\Gamma_2^- \times (0, T))} < C_8 \end{aligned}$$

with $C_i, i = 1, \dots, 8$, positive constants independent of ϵ .

Assume also that the initial and boundary data $f_\epsilon^0, f_{\epsilon 0}, g_\epsilon^0$, and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of ϵ . Finally assume that (48) holds. Then:

- (i) A subsequence of $w_\epsilon = \int_{\mathbb{R}} g_\epsilon dv$ (still denoted w_ϵ) converges in $L^1_{loc}(\Omega_g \times (0, T))$ to a function w .
- (ii) The strong convergence in L^1_{loc} is actually a strong convergence in L^1 .
- (iii) A subsequence of g_ϵ (still denoted g_ϵ) converges in $L^\infty(\Omega_g \times V \times (0, T))$ weak- \star to a function g and $g = \chi_w$.

The proof of point (i) in Theorem 4.5 relies on an extension to domains with boundaries of a compactness result introduced in [12]. This general compactness result is a variant of compactness results shown in [7]. The extension result is given in the following theorem.

Theorem 4.6. *Let $1 < p \leq 2$, let g be bounded in $L^p_{loc}(\Omega_g \times \mathbb{R}_v \times (0, T))$, let h belong to a compact set of $L^p(\Omega_g \times \mathbb{R}_v \times (0, T))$, and let $r \geq 0$. We assume that g satisfies*

$$(53) \quad \frac{\partial g}{\partial t} + \sum_{i=1}^d a_i(v) \frac{\partial g}{\partial x_i} = (-\Delta_{x,t} + 1)^{1/2} (-\Delta_v + 1)^{r/2} h \quad \text{in } \mathcal{D}',$$

where $a(\cdot) \in C^{s,\alpha}_{loc}$ with $s = r$, $\alpha = 1$ if r is an integer, $s = [r]$, and $\alpha = r - s$ if r is not an integer. Let $\psi \in L^{p'}(\mathbb{R}_v)$ with compact support. Finally, we assume that $a(\cdot)$ satisfies (48). Then, $\int_{\mathbb{R}} g \psi dv$ belongs to a compact set of $L^p_{loc}(\Omega_g \times (0, T))$.

The proof of this theorem relies on a slight modification of the proof of the related result for the domain $\mathbb{R}^d \times \mathbb{R}_v \times (0, \infty)$. The proof of the result on the domain $\mathbb{R}^d \times \mathbb{R}_v \times (0, \infty)$ is given in [12] and proved there; see also [7].

The proof of point (ii) in Theorem 4.5 relies on Theorem 4.4.

Proof of Theorem 4.5. (i) For the proof of point (i) of this theorem we follow [12]; see also [27] and refer to it for more details. We first notice that using Lemmas 2.1-2.2, g_ϵ is uniformly bounded in $L^q(\Omega_g \times V \times (0, T))$ for any $q \geq 1$. Second, we prove that we can write $\frac{1}{\epsilon}(\chi_{w_\epsilon(x,t)}(v) - g_\epsilon(x, v, t))$ as $\partial m_\epsilon / \partial v$ with m_ϵ a nonnegative measure on $\Omega_g \times \mathbb{R}_v \times (0, T)$ bounded independently of $\epsilon > 0$. By applying Sobolev embeddings with $p < (d+2)/(d+1)$, we conclude that m_ϵ is bounded in

$$W^{-s,p}(\Omega_g \times \mathbb{R}_v \times (0, T))$$

for $(d+2)/p' < s < 1$. Then $\partial m_\epsilon / \partial v$ can be written in the form of the right-hand side in (53) for $r > 1 + (d+2)/p'$. Taking $\psi(v) = \mathbb{I}_{[-R_\infty, R_\infty]}(v)$, we conclude the proof of the first assertion in the theorem.

(ii) We now proceed in proving the second assertion. Using Lemma 2.3 and the diagonal process to extract a further subsequence, if necessary, a subsequence of w_ϵ converges in L^∞ weak- \star to a function $w \in L^\infty(\Omega_g \times [0, T])$. Therefore, since $\Omega_g \times (0, T)$ is bounded, we can use Theorem 4.4 and conclude that the strong L^1_{loc} convergence of w_ϵ is actually a strong L^1 convergence.

(iii) Using Lemma 2.1 a subsequence of g_ϵ converges in L^∞ weak- \star to a function $g \in L^\infty(\Omega_g \times V \times [0, T])$. The fact that $g = \chi_w$ results from using equation (4), the strong L^1 convergence of χ_{w_ϵ} to χ_w (through (ii) above and the properties of the function χ), the L^∞ weak- \star convergence of g_ϵ , and the uniqueness of the limit. The theorem is now proved. \square

Next we give some convergence properties of g_ϵ and their relations to those of w_ϵ .

Theorem 4.7. *Assume that*

$$\begin{aligned} \|f_{\epsilon 0}\|_{L^\infty(\Gamma_1^- \times [0, T])} &< C_1, \quad \|g_\epsilon^0\|_{L^\infty(\Omega_g \times V)} < C_2, \quad \|f_\epsilon^0\|_{L^\infty(\Omega_l \times V)} < C_3, \\ \|g_{\epsilon 1}\|_{L^\infty(\Gamma_2^- \times [0, T])} &< C_4, \quad \|f_\epsilon^0\|_{L^1(\Omega_l \times V)} < C_5, \quad \|g_\epsilon^0\|_{L^1(\Omega_g \times V)} < C_6, \\ \|a(v) \cdot n_l f_{\epsilon 0}\|_{L^1(\Gamma_1^- \times (0, T))} &< C_7, \quad \|a(v) \cdot n_g g_{\epsilon 1}\|_{L^1(\Gamma_2^- \times (0, T))} < C_8 \end{aligned}$$

with $C_i, i = 1, \dots, 8$, positive constants independent of ϵ .

Assume also that the initial and boundary data $f_\epsilon^0, f_{\epsilon 0}, g_\epsilon^0$, and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of ϵ . Finally assume that (48) holds. Then

(i) For any real k ,

$$\int_V |g_\epsilon - \chi_k| dv - |w_\epsilon - k| \rightarrow_{\epsilon \rightarrow 0} 0 \text{ in } L^1_{loc}(\Omega_g \times (0, T)).$$

(ii) For any $b(\cdot) \in L^\infty(V)$,

$$\int_V b(v) |g_\epsilon - \chi_k| dv - \text{sgn}(w_\epsilon - k) \int_V b(v) (g_\epsilon - \chi_k) dv \rightarrow_{\epsilon \rightarrow 0} 0 \text{ in } L^1_{loc}(\Omega_g \times (0, T)).$$

(iii)

$$\overline{\int_V |g_\epsilon - \chi_k| dv} = \overline{|w_\epsilon - k|} = |w - k|.$$

(iv)

$$\begin{aligned} \overline{\int_V a_i(v) |g_\epsilon - \chi_k| dv} &= \overline{\text{sgn}(w_\epsilon - k) \int_V a_i(v) (g_\epsilon - \chi_k)} \\ &= \text{sgn}(w - k) (A_i(w) - A_i(k)) \\ &= \int_V a_i(v) |\chi_w - \chi_k|. \end{aligned}$$

Here the over bar notation indicates the L^∞ weak- \star limit.

Proof of Theorem 4.7. The proofs of (i) and (ii) are exactly the same as the proof of Lemma 3.4 of [16]. The proofs of (iii) and (iv) are similar to the proofs of (3.17), (3.18), and (3.19) of [16]. In [16], the authors make the assumption that a subsequence of w_ϵ (in their notations u_ϵ) converges strongly to w : $w_\epsilon \rightarrow w$ (in their notations $u_\epsilon \rightarrow u$; see (3.9) in [16]). Here we know from Theorem 4.5 that w_ϵ converges strongly in $L^1(\Omega_g \times V \times (0, T))$ to w . Hence in our case the use of (3.9) in [16] must be replaced by the use of Theorem 4.5. Also in our case the justification of the weak- \star L^∞ -limit of the various quantities (after extraction of appropriate subsequences, if necessary) results from Lemmas 2.1 and 2.3. \square

Next, we state a result about the existence of a solution to the kinetic formulation of the coupled system given in (34)-(42).

Theorem 4.8. *Assume that*

$$\begin{aligned} \|f_{\epsilon 0}\|_{L^\infty(\Gamma_1^- \times [0, T])} &< C_1, \quad \|g_\epsilon^0\|_{L^\infty(\Omega_g \times V)} < C_2, \quad \|f_\epsilon^0\|_{L^\infty(\Omega_l \times V)} < C_3, \\ \|g_{\epsilon 1}\|_{L^\infty(\Gamma_2^- \times [0, T])} &< C_4, \quad \|f_\epsilon^0\|_{L^1(\Omega_l \times V)} < C_5, \quad \|g_\epsilon^0\|_{L^1(\Omega_g \times V)} < C_6, \\ \|a(v) \cdot n_l f_{\epsilon 0}\|_{L^1(\Gamma_1^- \times (0, T))} &< C_7, \quad \|a(v) \cdot n_g g_{\epsilon 1}\|_{L^1(\Gamma_2^- \times (0, T))} < C_8 \end{aligned}$$

with $C_i, i = 1, \dots, 8$, positive constants independent of ϵ .

Assume also that the initial and boundary data $f_\epsilon^0, f_{\epsilon 0}, g_\epsilon^0$, and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of ϵ .

Finally assume that (48) hold and that as $\epsilon \rightarrow 0$,

$$(54) \quad \|g_\epsilon^0(\cdot, \cdot) - \chi_{w^0}(\cdot)(\cdot)\|_{L^1_{loc}(\Omega_g \times L^1(V))} \rightarrow 0,$$

$$(55) \quad f_\epsilon^0 \rightarrow f^0 \text{ strongly in } L^1(\Omega_l \times V),$$

$$(56) \quad a(v) \cdot n f_{\epsilon 0} \rightarrow a(v) \cdot n f_0 \text{ weakly in } L^1(\Gamma_1^- \times (0, T)),$$

$$(57) \quad a(v) \cdot n g_{\epsilon 1} \rightarrow a(v) \cdot n g_1 = a(v) \cdot n \chi_{w_1} \text{ strongly in } L^1(\Gamma_2^- \times (0, T)).$$

Then f_ϵ converges strongly in $L^1(\Omega_l \times V \times (0, T))$, as ϵ goes to 0, to f , and g_ϵ converges in $L^\infty(\Omega_g \times V \times (0, T))$ weak- \star to g , and (f, g) is a solution of the system (34)-(42) in the sense of Definition 3.2.

Remark 4.2. A fundamental point in Definitions 3.1 and 3.2 of solutions to the coupled systems is the precise statement of the interface conditions between the kinetic model and its hydrodynamic conservation laws limit. The rigorous derivation and justification of such interface conditions is given in the following theorems.

Theorem 4.9. For any $\psi_1 \in C_0^1(\bar{\Omega}_l \times V \times (0, T))$ and for any $k \in \mathbb{R}$, we have

$$(58) \quad \lim_{\epsilon \rightarrow 0} \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n \psi_1 |f_\epsilon - \chi_k| = \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n \psi_1 |f - \chi_k|.$$

Theorem 4.10. For any $\psi_2 \in C_0^1(\bar{\Omega}_g \times V \times (0, T))$, we have

$$(59) \quad \lim_{\epsilon \rightarrow 0} \int_{\Gamma_i^- \times (0, T)} a(v) \cdot n \psi_2 g_\epsilon = \int_{\Gamma_i^- \times (0, T)} a(v) \cdot n \psi_2 \chi_w.$$

The proofs of Theorems 4.9-4.10 will be given at the end of this section. The proof of Theorem 4.10 relies on a new regularity theory introduced in this paper; see also the author's papers [27, 25].

Remark 4.3. Theorem 4.8 is obtained under various assumptions including the assumption that the data $f_\epsilon^0, f_{\epsilon 0}, g_\epsilon^0$, and $g_{\epsilon 1}$ are compactly supported in v . In fact this theorem is also valid when these data are not necessarily compactly supported in v . The proof is based on a BV-regularization argument.

Proof of Theorem 4.8. Using Theorems 4.1, 4.3, and 4.5, a subsequence of f_ϵ, u_ϵ , and w_ϵ (still denoted f_ϵ, u_ϵ , and w_ϵ) converges strongly in L^1 resp. to f, u , and w . Now using Theorem 4.5 a subsequence of g_ϵ (still denoted g_ϵ) converges in L^∞ weak- \star to $g = \chi_w$. It is clear that

$$(f, g) \in (L^\infty([0, T]; L^1(\Omega_l \times V)) \cap L^\infty(\Omega_l \times [0, T]; L^1(V))) \\ \times (L^\infty([0, T]; L^1(\Omega_g \times V)) \cap L^\infty(\Omega_g \times (0, T); L^1(V))).$$

We shall first prove that the first assertion in Definition 3.2 is satisfied; that is, we wish to prove that Eq. (37) is satisfied. Using the above we know that $[\partial_t + a(v) \cdot \partial_x]g_\epsilon(x, v, t)$ converges to $[\partial_t + a(v) \cdot \partial_x]g(x, v, t)$ in the distribution sense. Proceeding as in the proof of Theorem 4.5 and following [12] (see also [27]) we can write $\frac{1}{\epsilon}(\chi_{w_\epsilon(x, t)}(v) - g_\epsilon(x, v, t))$ as $\partial m_\epsilon / \partial v$ with m_ϵ a nonnegative measure on $\Omega_g \times \mathbb{R}_v \times (0, T)$ bounded independently of $\epsilon > 0$ and which converges to the unique measure m given in (37); i.e.

$$(60) \quad [\partial_t + a(v) \cdot \partial_x]g(x, v, t) = \frac{\partial m}{\partial v} \text{ in } \Omega_g \times V \times (0, T).$$

Hence the equation (37) is satisfied. We now prove the second part in Definition 3.2. Using Theorem 2.2, we have

$$\begin{aligned} & - \int_{\Omega_g \times V \times (0, T)} (\partial_t + a(v) \cdot \partial_x)(\psi_2)|g_\epsilon - \chi_k| + \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n_g \psi_2 |f_\epsilon - \chi_k| \\ & + \int_{\Gamma_2^- \times (0, T)} a(v) \cdot n_g \psi_2 |g_{\epsilon 1} - \chi_k| \leq 0 \quad \forall \psi_2 \in C_0^1(\bar{\Omega}_g \times (0, T)), \psi_2 \geq 0, \forall k \in \mathbb{R}. \end{aligned}$$

Using Theorems 4.5, 4.7, and 4.9, and (57), and letting $\epsilon \rightarrow 0$ in the above inequality, we obtain

$$\begin{aligned} & - \int_{\Omega_g \times V \times (0, T)} (\partial_t + a(v) \cdot \partial_x)(\psi_2)|g - \chi_k| + \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n_g \psi_2 |f - \chi_k| \\ & + \int_{\Gamma_2^- \times (0, T)} a(v) \cdot n_g \psi_2 |\chi_{w_1} - \chi_k| \leq 0 \quad \forall \psi_2 \in C_0^1(\bar{\Omega}_g \times (0, T)), \psi_2 \geq 0, \forall k \in \mathbb{R} \end{aligned}$$

with $g = \chi_w$.

For the local problem (problem in Ω_l) we know that f satisfies (34); see Theorem 4.3. On the other hand f_ϵ satisfies for any $\psi_1 \in C_0^1(\bar{\Omega}_l \times V \times (0, T))$

$$\begin{aligned} & - \int_{\Omega_l \times V \times (0, T)} (\partial_t + a(v) \cdot \partial_x)(\psi_1)f_\epsilon + \int_{\Gamma_1^- \times (0, T)} a(v) \cdot n \psi_1 f_{\epsilon 0} \\ & + \int_{\Gamma_1^+ \times (0, T)} a(v) \cdot n \psi_1 f_\epsilon + \int_{\Gamma_i^- \times (0, T)} a(v) \cdot n \psi_2 g_\epsilon \\ (61) \quad & + \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n \psi_2 f_\epsilon = \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, T)} (\chi_{u_\epsilon} - f_\epsilon) \psi_1. \end{aligned}$$

Using Theorems 4.1 and 4.3, together with the properties of χ , the first term converges to

$$(62) \quad - \int_{\Omega_l \times V \times (0, T)} (\partial_t + a(v) \cdot \partial_x)(\psi_1)f$$

and the right side in (61) converges to

$$(63) \quad \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, T)} (\chi_u - f) \psi_1.$$

Using (56), the second term in (61) converges to

$$(64) \quad \int_{\Gamma_1^- \times (0, T)} a(v) \cdot n \psi_1 f_0.$$

Now let $\psi_1 \in C_0^1(\bar{\Omega}_l \times V \times (0, T))$ be such that $\psi_1 \equiv 0$ on $\Gamma_1 \times V \times (0, T)$ and on $\Gamma_i^- \times (0, T)$. Multiplying equation (34) by ψ_1 and integrating by parts, we obtain

$$\begin{aligned} & \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n \psi_1 f = \int_{\Omega_l \times V \times (0, T)} (\partial_t + a(v) \cdot \partial_x)(\psi_1)f \\ (65) \quad & + \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, T)} (\chi_u - f) \psi_1. \end{aligned}$$

On the other hand multiplying Eq. (1) by ψ_1 and integrating by parts, we obtain

$$(66) \quad \int_{\Gamma_i^+ \times (0,T)} a(v) \cdot n\psi_1 f_\epsilon = \int_{\Omega_l \times V \times (0,T)} (\partial_t + a(v) \cdot \partial_x)(\psi_1) f_\epsilon + \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0,T)} (\chi_{u_\epsilon} - f_\epsilon) \psi_1.$$

Thanks to (62) and (63), letting $\epsilon \rightarrow 0$ in (66) and using (65), we obtain

$$(67) \quad \lim_{\epsilon \rightarrow 0} \int_{\Gamma_i^+ \times (0,T)} a(v) \cdot n\psi_1 f_\epsilon = \int_{\Gamma_i^+ \times (0,T)} a(v) \cdot n\psi_1 f.$$

By a similar method, we obtain

$$(68) \quad \lim_{\epsilon \rightarrow 0} \int_{\Gamma_1^+ \times (0,T)} a(v) \cdot n\psi_1 f_\epsilon = \int_{\Gamma_1^+ \times (0,T)} a(v) \cdot n\psi_1 f.$$

Using the transmission condition (3) and Theorem 4.10, we obtain

$$(69) \quad \lim_{\epsilon \rightarrow 0} \int_{\Gamma_i^- \times (0,T)} a(v) \cdot n\psi_1 f_\epsilon = \int_{\Gamma_i^- \times (0,T)} a(v) \cdot n\psi_1 \chi_w.$$

Thus, letting $\epsilon \rightarrow 0$ in (61) and using (62), (64), (63), (67), (68), and (69), we obtain

$$(70) \quad \begin{aligned} & - \int_{\Omega_l \times V \times (0,T)} (\partial_t + a(v) \cdot \partial_x)(\psi_1) f + \int_{\Gamma_1^- \times (0,T)} a(v) \cdot n\psi_1 f_0 \\ & + \int_{\Gamma_1^+ \times (0,T)} a(v) \cdot n\psi_1 f + \int_{\Gamma_i^- \times (0,T)} a(v) \cdot n\psi_1 \chi_w \\ & + \int_{\Gamma_i^+ \times (0,T)} a(v) \cdot n\psi_1 f = \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0,T)} (\chi_u - f) \psi_1. \end{aligned}$$

Finally, thanks to the L^1 strong convergence of f_ϵ and assumption (55), f satisfies the initial condition in (42). Now set $w_\epsilon^0 = \int_V g_\epsilon^0$. Since g_ϵ^0 satisfies condition (54) and the support in x of g_ϵ^0 is included in a fixed compact set independent of ϵ and g_ϵ^0 is uniformly bounded in $L^\infty(\Omega_g \times V)$, we may use Theorem 4.4 to infer that g_ϵ^0 converges strongly to χ_{w^0} in $L^1(\Omega_g \times V)$. Thus, by the properties of χ , w_ϵ^0 converges strongly to w^0 in L^1 . Moreover, we know from Theorem 4.5 that w_ϵ strongly converges in L^1 to w . We then conclude that $w(\cdot, 0) = w^0(\cdot)$ and by the properties of χ and Theorem 4.5(iii), $g(\cdot, \cdot, 0) = \chi_{w^0(\cdot)}(\cdot)$. Hence, g satisfies the initial condition in (42). Therefore (f, w) is an entropic solution in the sense of Definition 3.2 to the system (34)-(42). The proof of the theorem is now complete. \square

Proof of Theorem 4.9. Let $\psi_1 \in C_0^1(\bar{\Omega}_l \times V \times (0, T))$ be such that $\psi_1 \equiv 0$ on $\Gamma_i^- \times (0, T)$ and $\psi_1 \equiv 0$ on $\Gamma_1 \times V \times (0, T)$. Using Theorem 2.2 we have

$$(71) \quad \begin{aligned} & \int_{\Gamma_i^+ \times (0,T)} a(v) \cdot n\psi_1 |f_\epsilon - \chi_k| \\ & = \int_{\Omega_l \times V \times (0,T)} (\partial_t + a(v) \cdot \partial_x)(\psi_1) |f_\epsilon - \chi_k| \\ & + \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0,T)} (\chi_{u_\epsilon} - f_\epsilon) \psi_1 \text{sign}(f_\epsilon - \chi_k) \quad \forall k \in \mathbb{R}. \end{aligned}$$

Using Theorems 4.1 and 4.3 together with the properties of χ , the right side in (71) converges as $\epsilon \rightarrow 0$ to

$$(72) \quad \begin{aligned} & \int_{\Omega_l \times V \times (0, T)} (\partial_t + a(v) \cdot \partial_x)(\psi_1)|f - \chi_k| \\ & + \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, T)} (\chi_u - f)\psi_1 \text{sign}(f - \chi_k) \quad \forall k \in \mathbb{R}. \end{aligned}$$

Since f satisfies Eq. (34) (Theorem 4.3) we can define the trace of f and apply Green's formula to equation (34) (Theorems 2.3 and 2.4). Proceeding as in the proof of Theorem 2.2 (see also [25]) we also have

$$(73) \quad \begin{aligned} & \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n \psi_1 |f - \chi_k| \\ & = \int_{\Omega_l \times V \times (0, T)} (\partial_t + a(v) \cdot \partial_x)(\psi_1)|f - \chi_k| \\ & + \frac{1}{\epsilon_l} \int_{\Omega_l \times V \times (0, T)} (\chi_u - f)\psi_1 \text{sign}(f - \chi_k) \quad \forall k \in \mathbb{R}. \end{aligned}$$

Therefore taking the limit as $\epsilon \rightarrow 0$ in (71) and using (72) and (73) we obtain

$$(74) \quad \lim_{\epsilon \rightarrow 0} \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n \psi_1 |f_\epsilon - \chi_k| = \int_{\Gamma_i^+ \times (0, T)} a(v) \cdot n \psi_1 |f - \chi_k|$$

which corresponds to (58). \square

Proof of Theorem 4.10. The proof is split into 5 steps.

Step 1. We give the equation satisfied by the function g .

We already know that g_ϵ is uniformly bounded in L^1 and L^∞ (Lemmas 2.1, 2.2, and 2.3) and the limit $g = \chi_w$ (see Theorem 4.5(iii)) is in L^1 . Moreover, $[\partial_t + a(v) \cdot \partial_x]g_\epsilon(x, v, t)$ converges to $[\partial_t + a(v) \cdot \partial_x]g(x, v, t)$ in the distribution sense. Proceeding as in [12] and [27], we can write $\frac{1}{\epsilon}(\chi_{w_\epsilon(x, t)}(v) - g_\epsilon(x, v, t))$ as $\partial m_\epsilon / \partial v$ and equation (4) in the form

$$(75) \quad [\partial_t + a(v) \cdot \partial_x]g_\epsilon(x, v, t) = \frac{\partial m_\epsilon}{\partial v} \quad \text{in } \Omega_g \times V \times (0, T)$$

with m_ϵ a nonnegative measure on $\Omega_g \times V \times (0, T)$ bounded independently of $\epsilon > 0$ and which converges to the unique measure m given by

$$(76) \quad [\partial_t + a(v) \cdot \partial_x]g(x, v, t) = \frac{\partial m}{\partial v} \quad \text{in } \Omega_g \times V \times (0, T)$$

and whose support in the variable $v \in V$ is included by Lemma 2.3 in a compact set of V $[-R_0, R_0]$ (since the supports of the measures m_ϵ are included by the same lemma in a fixed compact set independent of ϵ). In fact, we can deduce from the above equation and the form of g that the support in $v \in V$ of m is actually included in $[-\tilde{R}, \tilde{R}]$, where $\tilde{R} = \|w\|_\infty$. For more on this consult [12]. We shall take $R = \max(R_0, \tilde{R})$; then the support of m_ϵ , g_ϵ , χ_{w_ϵ} and m are all contained in the fixed compact set $[-R, R]$.

Step 2. We prove the existence of the normal trace of the flux $ga(v)$ on the boundary.

Let $\psi \in C_0^1(\bar{\Omega}_g \times (0, T))$ with $\psi \equiv 0$ on $\Gamma_2 \times (0, T)$ and

$$V^- = \{v \in V : a(v) \cdot n_g(x) > 0 \text{ for some } x \in \Gamma_i\}.$$

Notice that $\Gamma_i \times V^- = \Gamma_i^-$.

Let $\varphi \in C_0^1(V)$ with $\varphi \equiv 0$ on $V \setminus V^-$ be fixed. Let $F = (\int_V g\varphi, \int_V a_1(v)g\varphi, \dots, \int_V a_d(v)g\varphi)$. It is clear from the proof of Theorem 4.5 that $F \in L^\infty(\Omega_g \times (0, T))$. We shall prove that $\operatorname{div}_{t,x} F = \partial_t \int_V g\varphi + \sum_{i=1}^d \partial_{x_i} \int_V a_i(v)g\varphi$ is a bounded measure.

Let $\psi \in C_0^1(\Omega_g \times (0, T))$. Using Eq. (76), we obtain

$$(77) \quad \langle [\partial_t + a(v) \cdot \partial_x]g, \varphi \otimes \psi \rangle = - \int_{\Omega_g \times V \times (0, T)} \partial_v \varphi \psi dm.$$

Thus we have

$$(78) \quad \langle \operatorname{div}_{t,x} F, \psi \rangle = - \int \partial_v \varphi \psi dm.$$

Since m is a bounded measure on $\Omega_g \times V \times (0, T)$ and $\partial_v \varphi \in C_0(V)$, $\operatorname{div}_{t,x} F$ is also a bounded measure on $\Omega_g \times (0, T)$. Now $F \in L^\infty(\Omega_g \times [0, T])$ and $\operatorname{div}_{t,x} F$ is a bounded measure. Therefore, we can apply the Gauss-Green formula of Theorem 2.5 with $U = \Omega_g \times (0, T)$ and we obtain in particular

$$\begin{aligned} \langle F \cdot \nu|_{\partial U}, \psi \rangle &= \int_U \psi \operatorname{div}_{t,x} F + \int_U \nabla_{t,x} \psi \cdot F \quad \forall \psi \in C_0^1(\bar{\Omega}_g \times (0, T)) \\ &= \operatorname{ess\,limit}_{z \rightarrow 0} \int_{\partial U_z} \psi \circ \Psi_z^{-1} F \cdot \nu d\mathcal{H}^1. \end{aligned}$$

Here $\nabla_{t,x} = (\partial_t, \partial_x)$ and $\nu(u)$ is the unit outward normal to ∂U_z at $u \in \partial U_z$. This is equivalent to

$$\begin{aligned} & - \int_{\Omega_g \times V \times (0, T)} \partial_v \varphi \psi dm + \int_{\Omega_g \times V \times (0, T)} g\varphi [\partial_t + a(v) \cdot \partial_x] \psi \\ &= \operatorname{ess\,limit}_{z \rightarrow 0} \int_{\partial U_z} \psi \circ \Psi_z^{-1} F \cdot \nu d\mathcal{H}^1. \end{aligned}$$

Taking $\psi \in C_0^1(\bar{\Omega}_g \times (0, T))$ with $\psi \equiv 0$ on $\partial U \setminus (\Gamma_i \times (0, T))$, the above equations yield

$$\begin{aligned} (79) \quad & - \int \partial_v \varphi \psi dm + \int_{\Omega_g \times V \times (0, T)} g\varphi [\partial_t + a(v) \cdot \partial_x] \psi \\ &= \langle F \cdot \nu|_{\Gamma_i \times (0, T)}, \psi \rangle \\ &= \operatorname{ess\,limit}_{z \rightarrow 0} \int_{\Psi((\Gamma_i \times (0, T)) \times \{z\})} \psi \circ \Psi_z^{-1} F \cdot \nu d\mathcal{H}^1 \\ (80) \quad &= \operatorname{ess\,limit}_{z \rightarrow 0} \int_{\Psi((\Gamma_i \times (0, T)) \times \{z\}) \times V} \psi \circ \Psi_z^{-1} \varphi g \tilde{a}(v) \cdot \nu d\mathcal{H}^1, \end{aligned}$$

where $\tilde{a}(v) = (a_0(v), a(v)) = (1, a(v))$.

Step 3. We prove the convergence of the traces on $\Gamma_i^- \times (0, T)$.

Using the integral representation Theorem 2.1, we infer that g_ϵ converges L^∞ weakly star on $\Gamma_i^- \times (0, T)$ to a function $\tilde{g} \in L^\infty(\Gamma_i^- \times (0, T))$. Hence, we have

$$(81) \quad \lim_{\epsilon \rightarrow 0} \int_{\Gamma_i^- \times (0, T)} a(v) \cdot n \phi g_\epsilon = \int_{\Gamma_i^- \times (0, T)} a(v) \cdot n \phi \tilde{g}, \quad \forall \phi \in L^1(\Gamma_i^- \times (0, T)),$$

where we have also used Lemmas 2.3. In particular, we have

$$(82) \quad \lim_{\epsilon \rightarrow 0} \int_{\Gamma_i^- \times (0, T)} a(v) \cdot n \varphi \psi g_\epsilon = \int_{\Gamma_i^- \times (0, T)} a(v) \cdot n \varphi \psi \tilde{g}$$

$\forall \varphi \in C_0^1(V)$, with $\varphi \equiv 0$ on $V \setminus V^-$, and $\forall \psi \in C_0^1(\bar{\Omega}_g \times (0, T))$.

Step 4. Let $\psi \in C_0^1(\bar{\Omega}_g \times (0, T))$ with $\psi \equiv 0$ on $(\partial\Omega_g \times (0, T)) \setminus (\Gamma_i \times (0, T))$. It is clear that applying Theorem 2.4 to equation (4) for g_ϵ , we obtain

$$(83) \quad \begin{aligned} & \int_{\Omega_g \times V \times (0, T)} \varphi \psi [\partial_t + a(v) \cdot \partial_x] g_\epsilon + \int_{\Omega_g \times V \times (0, T)} g_\epsilon \varphi [\partial_t + a(v) \cdot \partial_x] \psi \\ &= - \int_{\Omega_g \times V \times (0, T)} \partial_v \varphi \psi dm_\epsilon + \int_{\Omega_g \times V \times (0, T)} g_\epsilon \varphi [\partial_t + a(v) \cdot \partial_x] \psi \end{aligned}$$

$$(84) \quad = \int_{\Gamma_i^- \times (0, T)} a(v) \cdot n \varphi \psi g_\epsilon.$$

In this step, we shall prove that the expression in (83) converges to the expression in (79). It is enough to prove such statement for $\psi \in C_0^1(\bar{\Omega}_g \times (0, T))$ of the form $\psi(x, t) = \tilde{\psi}(x)\phi(t)$ with $\tilde{\psi}(x) \in C_0^1(\bar{\Omega}_g)$ and $\phi(t) \in C_0^1(0, T)$.

We shall split the proof into six steps.

4.1. Let G denote the flux $G(x) = \int_{V \times (0, T)} \varphi(v)\phi(t)ga(v)dvdt$ on Ω_g . Eq. (76) yields

$$(85) \quad \partial_x G = - \int_{V \times (0, T)} \partial_v \varphi(v)\phi(t)dm + \int_{V \times (0, T)} \varphi(v)\partial_t \phi(t)g \text{ in } \mathcal{D}'(\Omega_g),$$

where $\varphi(v) \in C_0^1(V)$ with $\varphi \equiv 0$ on $V \setminus V^-$ and $\phi(t) \in C_0^1(0, T)$. Similarly, let G_ϵ denote the flux $G_\epsilon(x) = \int_{V \times (0, T)} \varphi(v)\phi(t)g_\epsilon a(v)dvdt$. By Lemma 2.2, G_ϵ is uniformly (in ϵ) bounded in $L^1(\Omega_g)$. Eq. (75) yields

$$(86) \quad \partial_x G_\epsilon = - \int_{V \times (0, T)} m_\epsilon \partial_v \varphi(v)\phi(t) + \int_{V \times (0, T)} \varphi(v)\partial_t \phi(t)g_\epsilon \text{ in } \mathcal{D}'(\Omega_g).$$

Now the first term in the right-hand side of (86) is a bounded measure thanks to the fact that $\phi(t)$ is of compact support in $t \in (0, T)$, $\partial_v \varphi(v)$ is of compact support in $v \in V$, and m_ϵ is a bounded measure. Thanks to Lemma 2.2, the second term is a function uniformly (in ϵ) bounded in $L^1(\Omega_g)$. Thus, the distributional derivative of G_ϵ is a bounded measure and hence G_ϵ is of bounded variation in Ω_g . In fact, the right-hand side of (86) is also equal to

$$\int_{V \times (0, T)} \partial_v m_\epsilon \varphi(v)\phi(t) + \int_{V \times (0, T)} \varphi(v)\partial_t \phi(t)g_\epsilon$$

which is a function in $L^1(\Omega_g)$ since $\partial_v m_\epsilon = \frac{1}{\epsilon}(\chi_{w_\epsilon} - g_\epsilon)$ and both χ_{w_ϵ} and g_ϵ are in L^1 (see Step 1). Then G_ϵ is a function of one variable (recall that we assumed $d = 1$ and $\Omega_g = (l, 1)$) whose distributional derivative is also a function. Therefore, G_ϵ is an absolutely continuous function [17].

4.2. We prove that $\partial_x G_\epsilon$ is uniformly bounded in $L^1(\Omega_g)$.

Using the fact that $\partial_x G_\epsilon$ is a function and (86) we have

$$(87) \quad \int_{\Omega_g} |\partial_x G_\epsilon| \leq \int_{\Omega_g \times V \times (0, T)} m_\epsilon |\partial_v \varphi(v) \phi(t)| + \int_{\Omega_g \times V \times (0, T)} |\varphi(v) \partial_t \phi(t) g_\epsilon|.$$

Using the positiveness and uniform boundedness of the measures m_ϵ , Lemma 2.2, and the definition of the functions $\varphi(v)$ and $\phi(t)$, the right-hand side in the inequality above is uniformly (in ϵ) bounded and so is the left-hand side.

4.3. Let P denote an extension operator from $W^{1,1}(\Omega_g)$ to $W^{1,1}(\mathbb{R})$. We shall prove that for any $\beta \in L^1(\mathbb{R})$ with compact support, $\beta \star \partial_x P G_\epsilon$ converges to $\beta \star \partial_x P G$ strongly in $L^1(\mathbb{R})$. Such an extension operator P exists [3]. Here we use the same operators P constructed in [3].

Let $H = PG$ and $H_\epsilon = P G_\epsilon$ (from Step 4.1 above we know that G_ϵ is in $W^{1,1}(\Omega_g)$). Using Steps 4.1 and 4.2, the definition of P , and its properties, we know that $\partial_x H_\epsilon$ is uniformly bounded in $L^1(\mathbb{R})$ and its support is included in a fixed compact set $[-R_1, R_1]$ for some $R_1 > 0$ independent of ϵ . Then the set

$$\mathcal{B} = \{\partial_x H_\epsilon, \epsilon > 0\}$$

is a bounded subset of $L^1(\mathbb{R})$. Let $\beta(x) \in L^1(\mathbb{R})$ with compact support $K_\beta \subset \mathbb{R}$. We shall prove that the set

$$\mathcal{F} = \{\beta \star \partial_x H_\epsilon, \epsilon > 0\}$$

is relatively compact in $L^1(\omega)$, for all ω open bounded set of \mathbb{R} . The notation \star refers to the convolution of two functions.

Let ω be an open bounded set of \mathbb{R} such that ω contains $\overline{[-R_1, R_1] + K_\beta}$. This implies that ω contains the support of the convolution $\beta \star \partial_x H_\epsilon$. Then we know that (a subsequence of) $\beta \star \partial_x H_\epsilon \rightarrow_{\epsilon \rightarrow 0} h^{\beta, \omega}$ in $L^1(\omega)$ for some function $h^{\beta, \omega} \in L^1(\omega)$. Because of the restriction above on the supports and the definition of ω , the convergence takes place actually in $L^1(\mathbb{R})$. We then simply denote $h^{\beta, \omega}$ by h^β .

We now wish to prove that $h^\beta = \beta \star \partial_x H$ in the distribution sense.

Let $\alpha(x) \in \mathcal{D}(\mathbb{R})$ be such that $\alpha \equiv 1$ on $[-R_1, R_1]$. Let $\eta(x) \in \mathcal{D}(\mathbb{R})$; we have

$$(88) \quad \begin{aligned} \langle (\beta \star \partial_x H_\epsilon)_x, \eta(x) \rangle &= \langle \beta_y \otimes (\partial_x H_\epsilon)_x, \eta(y+x) \rangle \\ &= \langle \beta_y \otimes (\partial_x H_\epsilon)_x, (\alpha(x) \eta(y+x)) \rangle \\ &= \int_{y \in \mathbb{R}} \beta(y) \int_{\mathbb{R}} \alpha(x) \eta(y+x) \partial_x H_\epsilon dx dy \\ &= - \int_{y \in K_\beta} \beta(y) \langle H_\epsilon, \partial_x (\alpha(x) \eta(y+x)) \rangle dy. \end{aligned}$$

The notation $\langle T_x, \eta(y+x) \rangle$ above simply refers to the fact that the distribution T is applied to the function $\theta(x) = \eta(y+x)$. It is not to be confused with the partial derivative with respect to x , which we denote throughout this paper by $\partial_x T$.

Thanks to the uniform boundedness of G_ϵ in $L^1(\Omega_g)$ (see the beginning of Step 4.1) and the properties of the extension operator P , the expression

$$\langle H_\epsilon, \partial_x (\alpha(x) \eta(y+x)) \rangle = \int H_\epsilon \partial_x (\alpha(x) \eta(y+x)) dx$$

is uniformly (in ϵ) bounded. On the other hand Theorem 4.5 shows that

$$\begin{aligned} \int_{\Omega_g} G_\epsilon \partial_x(\alpha(x)\eta(y+x))dx &= \int_{\Omega_g \times V \times (0,T)} g_\epsilon a(v)\varphi(v)\phi(t)\partial_x(\alpha(x)\eta(y+x))dvdt dx \\ &\xrightarrow{\epsilon \rightarrow 0} \int_{\Omega_g \times V \times (0,T)} ga(v)\varphi(v)\phi(t)\partial_x(\alpha(x)\eta(y+x))dvdt dx \\ &= \int_{\Omega_g} G \partial_x(\alpha(x)\eta(y+x))dx. \end{aligned}$$

Hence using the explicit definition of the extension operator P , we have

$$\langle H_\epsilon, \partial_x(\alpha(x)\eta(y+x)) \rangle = \int_{\mathbb{R}} H_\epsilon \partial_x(\alpha(x)\eta(y+x))dx$$

converges as $\epsilon \rightarrow 0$ to

$$\int_{\mathbb{R}} H \partial_x(\alpha(x)\eta(y+x))dx = \langle H, \partial_x(\alpha(x)\eta(y+x)) \rangle.$$

Therefore, since $\beta \in L^1(\mathbb{R})$ with compact support, we can use the dominated convergence theorem to conclude that the right-hand side in (88) converges to

$$\begin{aligned} - \int_{y \in K_\beta} \beta(y) \langle H, \partial_x(\alpha(x)\eta(y+x)) \rangle dy &= \int_{y \in \mathbb{R}} \beta(y) \langle \partial_x H, (\alpha(x)\eta(y+x)) \rangle dy \\ (89) \qquad \qquad \qquad &= \langle (\beta \star \partial_x H)_x, \eta(x) \rangle. \end{aligned}$$

On the other hand, using the distributional definition of the convolution for $\beta \star \partial_x H_\epsilon$ and the fact that $\beta \star \partial_x H_\epsilon$ converges to h^β in $L^1(\mathbb{R})$, together with (89), we obtain

$$(90) \qquad \qquad \qquad \langle h^\beta, \eta \rangle = \langle \beta \star \partial_x H, \eta \rangle.$$

This shows that the convolution (of two distributions) $\beta \star \partial_x H$ is a function in $L^1(\mathbb{R})$ equal to h^β . Hence for any $\beta \in L^1(\mathbb{R})$ with compact support we have proved that $\beta \star \partial_x H_\epsilon$ converges in $L^1(\mathbb{R})$ to $h^\beta = \beta \star \partial_x H$.

4.4. We prove that the distribution $\partial_x H$ is actually a measure.

Step 4.3 above implies in particular that for any $\beta \in L^1(\mathbb{R})$ with compact support, $\beta \star \partial_x H$ is a function in $L^1(\mathbb{R})$. Hence $\beta \star \partial_x H$ defines a measure on \mathbb{R} .

We shall prove that the distribution $\partial_x H$ is actually a measure. This results from the fact that if T is a distribution such that $\forall \beta \in L^1(\mathbb{R})$ with compact support, $T \star \beta$ is a measure, then T is itself a measure [17].

4.5. We prove that

$$(91) \qquad \qquad \qquad \lim_{\epsilon \rightarrow 0} \int_{\Omega_g} \tilde{\psi}(y) (\partial_x G_\epsilon(y) - \partial_x G(y)) dy = 0$$

$\forall \tilde{\psi}$ continuous bounded function on Ω_g .

In the previous steps 4.3 and 4.4, we proved that $h^\beta = \beta \star \partial_x H$ in the classical sense and that $\partial_x H$ is a measure. Denote μ as the measure $\mu = \partial_x H$. Then we know from the definition of the convolution of a function and a measure that the function $\beta \star \mu(x) = \beta \star \partial_x H$ is given by

$$\beta \star \mu(x) = \int_{\mathbb{R}} \beta(x-y) d\mu(y) = \int_{\mathbb{R}} \beta(x-y) \partial_x H(y) dy.$$

(Recall that β is of compact support.) In the previous Steps 4.3 and 4.4, we proved that for any $\beta \in L^1(\mathbb{R})$ with compact support, we have $\beta \star \partial_x H_\epsilon \rightarrow_{\epsilon \rightarrow 0} h^\beta$ in $L^1(\mathbb{R})$. In other words, we have

$$\begin{aligned}
 & \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} |\beta \star \partial_x H_\epsilon - h^\beta| dx \\
 &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} |\beta \star \partial_x H_\epsilon - \beta \star \partial_x H| dx \\
 &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \beta(x-y) \partial_x H_\epsilon(y) dy - \int_{\mathbb{R}} \beta(x-y) \partial_x H(y) dy \right| dx \\
 (92) \quad &= 0.
 \end{aligned}$$

Then

$$(93) \quad \lim_{\epsilon \rightarrow 0} \left(\int_{\mathbb{R}} \beta(x-y) \partial_x H_\epsilon(y) dy - \int_{\mathbb{R}} \beta(x-y) \partial_x H(y) dy \right) = 0, \text{ for a.e. } x \in \mathbb{R}.$$

Let $x = \gamma$ be a point at which such limit exists and is 0. Let $\tilde{\psi}(y)$ be a continuous bounded function on $\bar{\Omega}_g = [l, 1]$. Let $\beta(z) = \tilde{\psi}(\gamma - z)$ for $(\gamma - z) \in [l, 1]$ and $\beta(z) = 0$ otherwise. Then $\beta \in L^1(\mathbb{R})$ and has a compact support. Therefore, plugging this β in (93), we obtain

$$(94) \quad \lim_{\epsilon \rightarrow 0} \int_{\Omega_g} \tilde{\psi}(y) (\partial_x H_\epsilon(y) - \partial_x H(y)) dy = 0.$$

Now the definition of the extension operator [3] yields $\partial_x H_\epsilon(y) = \partial_x G_\epsilon(y)$ and $\partial_x H(y) = \partial_x G(y)$ on Ω_g . Thus, we obtain (91).

4.6. Now combining Step 4.5, Theorem 4.5(iii), and (85)-(86), we conclude that

$$\begin{aligned}
 & \lim_{\epsilon \rightarrow 0} \int_{\Omega_g \times V \times (0, T)} \tilde{\psi}(y) \partial_v \varphi(v) \phi(t) dm_\epsilon = \int_{\Omega_g \times V \times (0, T)} \tilde{\psi}(y) \partial_v \varphi(v) \phi(t) dm \\
 (95) \quad & \forall \tilde{\psi} \text{ continuous bounded function on } \Omega_g.
 \end{aligned}$$

In particular, (95) is valid for any function $\tilde{\psi} \in C_0^1(\bar{\Omega}_g)$. Using this, we conclude that the expression in (83) converges to the expression in (79) for any $\varphi(v) \in C_0^1(V)$ with $\varphi \equiv 0$ on $V \setminus V^-$ and any $\psi \in C_0^1(\bar{\Omega}_g \times (0, T))$ of the form $\psi(x, t) = \tilde{\psi}(x) \phi(t)$. By the remark at the beginning of Step 4, this convergence takes place for all functions $\varphi(v) \in C_0^1(V)$ with $\varphi \equiv 0$ on $V \setminus V^-$ and all functions $\psi(x, t) \in C_0^1(\bar{\Omega}_g \times (0, T))$. This concludes the proof of our claim.

Step 5. Here we conclude the proof.

Using the above convergence in (84) and (80), we obtain

$$\begin{aligned}
 (96) \quad & \lim_{\epsilon \rightarrow 0} \int_{\Gamma_i^- \times (0, T)} a(v) \cdot n \varphi \psi g_\epsilon \\
 &= \lim_{\epsilon \rightarrow 0} \int_{\Gamma_i \times V^- \times (0, T)} a(v) \cdot n \varphi \psi g_\epsilon \\
 &= \text{ess } \lim_{z \rightarrow 0} \int_{\Psi((\Gamma_i \times (0, T)) \times \{z\})} \psi \circ \Psi_z^{-1} F \cdot \nu d\mathcal{H}^1 \\
 &= \text{ess } \lim_{z \rightarrow 0} \int_{\Psi((\Gamma_i \times (0, T)) \times \{z\}) \times V^-} \psi \circ \Psi_z^{-1} \varphi g \tilde{a}(v) \cdot \nu d\mathcal{H}^1.
 \end{aligned}$$

Since $\varphi \in C_0^1(V)$ and $\psi \in C_0^1(\bar{\Omega}_g \times (0, T))$ are arbitrary functions with $\varphi \equiv 0$ on $V \setminus V^-$ and $\psi \equiv 0$ on $\partial(\Omega_g \times (0, T)) \setminus (\Gamma_i \times (0, T))$, (96) and (82) show that \tilde{g} is the weak trace of g on $\Gamma_i^- \times (0, T) = \Gamma_i \times V^- \times (0, T)$. Moreover, this weak trace satisfies (81), which therefore yields the theorem. \square

5. ANALYSIS OF THE COUPLED KINETIC/CONSERVATION LAWS SYSTEM

In this final section, we state a result about the existence of an entropic solution to the coupled kinetic/conservation laws system (8)-(13) in the sense of Definition 3.1.

Theorem 5.1. *Assume that*

$$\begin{aligned} \|f_{\epsilon 0}\|_{L^\infty(\Gamma_1^- \times [0, T])} &< C_1, \quad \|g_\epsilon^0\|_{L^\infty(\Omega_g \times V)} < C_2, \quad \|f_\epsilon^0\|_{L^\infty(\Omega_l \times V)} < C_3, \\ \|g_{\epsilon 1}\|_{L^\infty(\Gamma_2^- \times [0, T])} &< C_4, \quad \|f_\epsilon^0\|_{L^1(\Omega_l \times V)} < C_5, \quad \|g_\epsilon^0\|_{L^1(\Omega_g \times V)} < C_6, \\ \|a(v) \cdot n_l f_{\epsilon 0}\|_{L^1(\Gamma_1^- \times (0, T))} &< C_7, \quad \|a(v) \cdot n_g g_{\epsilon 1}\|_{L^1(\Gamma_2^- \times (0, T))} < C_8 \end{aligned}$$

with $C_i, i = 1, \dots, 8$, positive constants independent of ϵ .

Assume also that the initial and boundary data $f_\epsilon^0, f_{\epsilon 0}, g_\epsilon^0$, and $g_{\epsilon 1}$ are compactly supported in $v \in V$ with supports included in a fixed compact set independent of ϵ . Finally assume that (48) holds and that as $\epsilon \rightarrow 0$,

$$\begin{aligned} \|g_\epsilon^0(\cdot, \cdot) - \chi_{w^0}(\cdot)(\cdot)\|_{L_{loc}^1(\Omega_g \times L^1(V))} &\rightarrow 0, \\ f_\epsilon^0 &\rightarrow f^0 \text{ strongly in } L^1(\Omega_l \times V), \\ a(v) \cdot n_l f_{\epsilon 0} &\rightarrow a(v) \cdot n_l f_0 \text{ weakly in } L^1(\Gamma_1^- \times (0, T)), \\ a(v) \cdot n_g g_{\epsilon 1} &\rightarrow a(v) \cdot n_g g_1 = a(v) \cdot n \chi_{w_1} \text{ strongly in } L^1(\Gamma_2^- \times (0, T)). \end{aligned}$$

Then there exists an entropic solution to the system (8)-(13) in the sense of Definition 3.1.

Remark 5.1. Following Remark 4.3, Theorem 4.8 is valid also when the initial and boundary data are no longer compactly supported in v . Therefore by the use of Theorem 3.1 on the equivalence between the kinetic and the entropic formulations and Theorem 4.8, Theorem 5.1 is also valid when the boundary and initial data are not necessarily compactly supported in v .

Proof of Theorem 5.1. This theorem is an immediate consequence of Theorem 4.8 and the kinetic formulation of the coupled system (8)-(13) given in Theorem 3.1. \square

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